Honors - Economic Analysis III

Lecture 4: Neoclassical model - Decentralization

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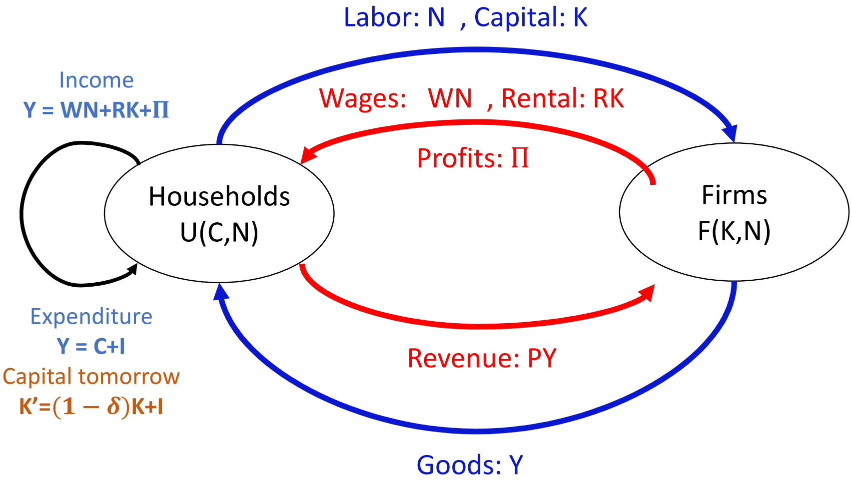
This lecture

- Decentralize the neo-classical model
- Imbed in a competitive two-sector economy
  1. Assumptions
  2. Optimality conditions
  3. Definition of a competitive equilibrium
- Welfare Theorems
- Next: Add government expenditure, taxes

Structure of the macroeconomy

Supply and demand of labor, capital, goods.



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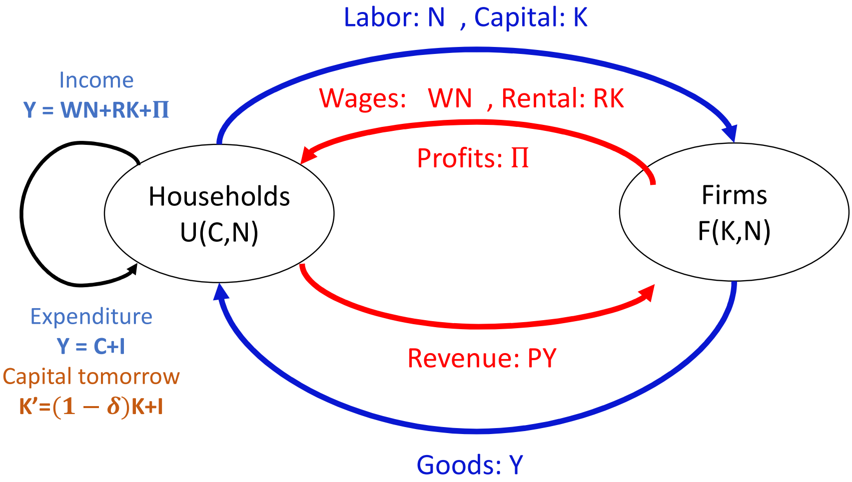

Environment - Centralized

- *Time* - Discrete $t = 0, 1, 2 \dots$
- *Agents* - Representative household with N workers
- *Goods* - One good can either be used for consumption or investment

$$c_t + i_t = y_t$$

- *Preferences* - Utility of the household at date 0 is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \underbrace{\beta \in (0,1)}_{\text{Rate of time preference}}$$

- *Technology* - Constant returns to scale production technology

$$y_t = Af(k_t) = Ak_t^{\alpha}.$$

Capital depreciates at rate δ

$$k_{t+1} = (1-\delta)k_t + i_t \quad , \quad \delta \in [0,1] \quad , \quad k_0 > 0$$

Environment - Decentralized

- *Time* - Discrete $t = 0, 1, 2 \ldots$
- *Agents* - Two types of agents
  1. Representative household with N workers   i.e. continuum of identical households
  2. Representative firm   i.e. continuum of identical firms
- *Goods* - One good can either be used for consumption or investment
- *Assets* - Households hold bonds B_t, which earn return $(1 + r_t)$. $B_0 \geq 0$.
- *Endowments* - Household owns all capital $K_0 > 0$ which it rents to firms at rate R_t.
- *Preferences* - Household utility of at date 0 is $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $c_t = C_t/N$
- *Technology* - Firms operate CRS production technology $y_t = Af(k_t)$. Capital depreciates at rate δ.
- *Markets* - **Competitive** markets: (i) goods $(P_t = 1)$, (ii) bonds (r_t), (iii) capital (R_t)

Problems - Household and Firm

- Household: Taking $\{W_t, R_t, r_t, \Pi_t\}_{t=0}^{\infty}$ as given—chooses sequences of $\{C_t, K_{t+1}\}_{t=0}^{\infty}$

$$U_0 = \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(C_t/N)$$

subject to the series of constraints

$$C_t + K_{t+1} + B_{t+1} \leq (1-\delta)K_t + W_t N + R_t K_t + (1+r_t)B_t + \Pi_t \quad , \quad t = 0, 1, 2, \ldots$$

and initial conditions

$$K_0 > 0 \, , \; B_0 \geq 0$$

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and initial conditions

$$K_0 > 0\,,\ B_0 \geq 0$$

- Firms: Each period t, taking W_t and R_t as given each firm chooses N_t, K_t

$$\Pi_t = \max_{K_t, N_t} F(K_t, N_t) - W_t N_t - R_t K_t$$

Problems - Household and Firm

- Household: Taking $\{W_t, R_t, r_t, \pi_t\}_{t=0}^{\infty}$ as given—chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$

$$U_0 = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the series of constraints

$$c_t + k_{t+1} + b_{t+1} \leq (1-\delta)k_t + W_t + R_t k_t + (1+r_t)b_t + \pi_t \quad , \quad t = 0, 1, 2, \ldots$$

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$$k_0 > 0 \, , \; b_0 \geq 0$$

- Firms: Each period t, taking W_t and R_t as given each firm chooses k_t

$$\pi_t = \max_{k_t} \; f(k_t) - W_t - R_t k_t$$

Household - Lagrangean

- Constrained optimization problem

$$
\begin{aligned}
\mathcal{L} = \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& + \sum_{t=0}^{\infty} \theta_t \Big[(1-\delta)k_t + W_t + R_t k_t + (1+r_t)b_t + \pi_t - c_t - k_{t+1} - b_{t+1} \Big]
\end{aligned}
$$

Household - Lagrangean

- Constrained optimization problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \theta_t \Big[(1-\delta)k_t + W_t + R_t k_t + (1+r_t)b_t + \pi_t - c_t - k_{t+1} - b_{t+1}\Big]$$

- First order necessary conditions

$$\begin{aligned} c_t: \quad 0 &= \beta^t u'(c_t) - \theta_t \\ k_{t+1}: \quad 0 &= -\theta_t + \theta_{t+1}\left[R_{t+1} + (1-\delta)\right] \\ b_{t+1}: \quad 0 &= -\theta_t + \theta_{t+1}(1 + r_{t+1}) \end{aligned}$$

Household - Lagrangean

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- Combining conditions

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})\left[R_{t+1} + (1-\delta)\right] \\ r_t &= R_t - \delta \end{aligned}$$

Household - No arbitrage

$$r_t = R_t - \delta$$

- Result - Equilibrium return on bonds is equal to the *net* return on capital

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- $r_t < R_t - \delta$
  - At $t-1$: Short 1 good (sell $b_t < 0$). Invest in one unit of capital k_t.
  - At t: Earn $R_t - \delta$ on k_t, use this to repay r_t and strictly increase goods in period t

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- $r_t > R_t - \delta$
  - At $t-1$: ... Homework
  - At t: ... Homework

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- $r_t > R_t - \delta$
  - At $t-1$: ... Homework
  - At t: ... Homework

- Measurement - We can now measure R_t (which is hard to observe) through $r_t + \delta$ (which are easier to observe). Can use this to infer β!

Household - Transversality conditions

- Similar logic to the centralized problem
- Now two conditions
- Given the constraint on $k_{t+1} \geq 0$ (which comes from physical nature of capital) we can show that

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

- But no such similar condition on bonds! $b_{t+1} \lesseqgtr 0$
- Optimality implies that the household will not die with positive bond holdings

$$\lim_{T \to \infty} \beta^T u'(c_T) b_{T+1} \leq 0$$

- What stops the household from setting b_{T+1} infinitely negative?

Household - Transversality conditions

- Need an additional assumption!

- Households can't run Ponzi schemes
  - In period 0, borrow $-b_1$, increase c_1 by b_1.
  - Then borrow $b_2 = -(1+r_1)b_1$ $\implies$ $b_2/(1+r_1) < 0$
  - Then borrow $b_3 = -(1+r_2)(1+r_1)b_1$ $\implies$ $b_3/(1+r_1)(1+r_2) < 0$
  - Then ... *keep on rolling this over*

- Restrict $\{b_t\}_{t=0}^{\infty}$ such that for all t

$$\lim_{T\to\infty} \frac{b_{t+1+T}}{\prod_{s=1}^{T}(1+r_{t+s})} \geq 0 \quad \implies \quad \lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} \geq 0$$

- *No Ponzi condition: The present discounted value of future debt must be positive*

- Show that $\implies$ is true under the household's Euler equation.

Household - Transversality conditions

- Similar logic to the centralized problem
- Now two conditions
- Given the constraint on $k_{t+1} \geq 0$ we can show that

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

- Combining (i) Optimality, and (ii) No-Ponzi conditions

$$\lim_{T \to \infty} \beta^T u'(c_T) b_{T+1} = 0$$

Firm - Static optimization

- Unconstrained optimization problem

$$\max_{K_t, N_t} \; \Pi_t = F(K_t, N_t) - W_t N_t - R_t K_t$$

- First order necessary conditions

$$\begin{aligned} K_t: \quad 0 &= F_K(K_t, N_t) - R_t \\ N_t: \quad 0 &= F_N(K_t, N_t) - W_t \end{aligned}$$

- ... in per-worker form

$$\begin{aligned} K_t: \quad 0 &= f'(k_t) - R_t \\ N_t: \quad 0 &= f(k_t) - f'(k_t)k_t - W_t \end{aligned}$$

Show: $\frac{\partial}{\partial N}\left\{ NF\left(\frac{K}{N}, 1\right)\right\} = f(k) - f'(k)k$

- Combining conditions

$$\begin{aligned} R_t &= f'(k_t) \\ W_t &= f(k_t) - f'(k_t)k_t \end{aligned}$$

Equilibrium

Definition - A *Competitive Equilibrium* for an economy with initial conditions k_0, b_0 is

- An *allocation* $\{c_t, b_{t+1}, k_{t+1}, \pi_t\}_{t=0}^{\infty}$
- Sequence of *prices* $\{W_t, R_t, r_t\}_{t=0}^{\infty}$

such that

1. **Optimality**
   - Given $\{W_t, R_t, r_t, \pi_t\}_{t=0}^{\infty}$, k_0, b_0, the allocation $\{c_t, b_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ maximizes the utility of the household.
   - Given $\{W_t, R_t, r_t\}_{t=0}^{\infty}$, the allocation $\{k_t\}_{t=0}^{\infty}$ maximizes profits of the firm, and delivers $\{\pi_t\}_{t=0}^{\infty}$
2. **Market clearing**
   - Labor - $N_t = \overline{N}$
   - Bonds - $b_t = 0$ *Bonds are in 'zero net-supply'*
   - Capital - Demand for capital by firms is equal to the supply of capital by households
   - Goods - $y_t = c_t + i_t$ where $i_t = k_{t+1} - (1 - \delta)k_t$

Welfare

- Household Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[R_{t+1} + (1-\delta)\right] \quad \text{, using } R_{t+1} = f'(k_{t+1}) \text{ from firm optimality}$$

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta)\right]$$

- Goods market clearing

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

Result

- The allocations of the competitive equilibrium and centralized economy coincide.

First and second welfare theorems

1. The competitive equilibrium is Pareto efficient
2. An allocation of the centralized economy can be obtained in the competitive equilibrium

Alternative decentralization I

- Household saves through bank deposits D_t. A representative bank holds and rents K_t to firms. $N = 1$
- The household budget constraint is

$$C_t + D_{t+1} + Q_t X_{t+1} \leq (1 + r_t) D_t + W_t N + \Big(Q_t + Div_t\Big) X_t$$

where $X_t \in [0, 1]$ is the number of shares that the household holds in the banking sector, and Div_t is the banking sector's dividend. Share price is Q_t.

- A bank chooses $\{D_t, K_t\}_{t=0}^{\infty}$ to maximize discounted dividends $\sum_{t=0}^{\infty} \mathcal{S}_t Div_t$

where bank dividends are cash-flows plus deposit growth net of investment:

$$Div_t = \Big[R_t K_t - r_t D_t\Big] + \Big[D_{t+1} - D_t\Big] - \Big[K_{t+1} - (1 - \delta) K_t\Big]$$

- Homework - What is the equilibrium deposit rate r_t and share price Q_t?
- Homework - Write $\mathcal{S}_t$ in terms of equilibrium prices **Hint:** Write $\mathcal{S}_{t+1} = \Psi_t \mathcal{S}_t$, where Ψ_t depends on prices. This implies $\mathcal{S}_{t+1} = \prod_0^t \Psi_t \, \mathcal{S}_0$. In words, what problem is the bank solving?
- Clue 1 - Banks are competitive. This implies that *in equilibrium* dividends are $Div_t = 0 \, \forall t$
- Clue 2 - *In equilibrium* the household holds all bank shares $X_t = 1$

Alternative decentralization II

- Household holds shares in a stock market of firms. Firms hold capital and invest. Let $N = 1$.
- The household budget constraint is

$$C_t + P_t S_{t+1} + B_{t+1} \leq W_t N + \Big(P_t + D_t\Big) S_t + (1 + r_t) B_t$$

  where $S_t \in [0, 1]$ is the number of shares that the household holds, and D_t is dividends per share. The price of a share is P_t.
- A firm chooses $\{K_t\}_{t=0}^{\infty}$ to maximize discounted dividends $\sum_{t=0}^{\infty} \mathcal{S}_t D_t$ where dividends are sales net of labor costs and investment:

$$D_t = F(K_t, N_t) - W_t N_t - \Big[K_{t+1} - (1 - \delta) K_t\Big]$$

- Homework - Show that, in equilibrium, the firm maximizes its share price, which is its present discounted value of dividends, discounted at the interest rate on bonds

$$P_0 = \max_{\{K_{t+1}, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{D_t}{\prod_{s=1}^{t}(1 + r_s)}$$

- Clue 1 - Show that $\mathcal{S}_t = \beta^t u'(C_t) = 1 / \left(\prod_{s=1}^{t}(1 + r_s)\right)$.
- Clue 2 - Combine Euler equations to write P_0 as a function of $\{P_1, r_1, D_1\}$ then iterate this forward

END