Honors - Economic Analysis III

Lecture 11: Idiosyncratic risk and complete markets

Simon Mongey

Winter, 2021

This lecture

- Up to now - Aggregate risk
- Last three classes - *Idiosyncratic risk*
- Welfare results in economies with *complete markets*
- Arrow securities, insurance in equilbirium
- PS8 - Solving simple complete markets models (I'll put it online but not due)
- Final class - Idiosyncratic risk + *Incomplete* markets

Environment

- *Time* - Discrete $t = 0, 1, 2 \ldots$
- ***Agents* - Set of individuals** $i \in \{1, 2, \ldots, N\}$
- *States* - Economy can be in finite states $s \in \{s_1, \ldots, s_S\}$. Prob $\pi(s^t)$

  Assume that s_t is *Markov* such that $\pi(s_{t+1}|s^t) = \pi(s_{t+1}|s_t)$.
- *Endowments* - Each individual's finite endowment depends only on s_t: $y^i(s_t) \in [0, \infty)$
- *Goods* - One perishable consumption good
- *Preferences* - Utility of each individual household at date 0 is:

$$U^i(s_0) = \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\left(c_t^i(s^t)\right) \quad , \quad \beta \in (0,1)$$

Note that individuals $i = 1$ and $i = 2$ might have different flow utility: $u^1(c) \neq u^2(c)$.

Efficient allocation

- We already have (almost) all the ingredients we need to write out the planner's problem and study the *efficient allocation* of the economy.
- In order to specify the objective function of the planner, though, we need to know how the planner *weights* each individual
- New - Let θ^i be a *Pareto weight* that the planner puts on individual i
- The *efficient allocation* is the allocation $\{c_t^i(s^t)\}_{i,t,s^t}$ that solves the following problem

$$\max_{c_t^i(s^t)} \sum_i \theta^i \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\Big(c_t^i(s^t)\Big)$$

subject to the resource constraint in all states

$$\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t) \qquad , \qquad \text{for all } t, s^t$$

- Eg: If $\theta^i = 1$ for all i, then we have equal weights. This is called a *utilitarian welfare function*

Planner's problem

- Write this as a Lagrangean with a multiplier $\lambda_t(s^t)$ on every resource constraint

$$\mathcal{L} = \max_{c_t^i(s^t)} \sum_i \theta^i \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\left(c_t^i(s^t)\right) - \sum_{t=0}^{\infty} \sum_{s^t|s_0} \lambda_t(s^t) \left[\sum_i y^i(s_t) - \sum_i c_t^i(s^t)\right]$$

- First order conditions for $c_t^i(s^t)$ and $c_t^j(s^t)$ for two individuals i and j

$$\begin{aligned} \beta^t \pi(s^t)\theta^i u^{i\prime}\left(c_t^i(s^t)\right) &= \lambda_t(s^t) \\ \beta^t \pi(s^t)\theta^j u^{j\prime}\left(c_t^j(s^t)\right) &= \lambda_t(s^t) \end{aligned}$$

- Combining these

$$\frac{u^{i\prime}\left(c_t^i(s^t)\right)}{u^{j\prime}\left(c_t^j(s^t)\right)} = \frac{\theta^j}{\theta^i}$$

- Result - *In the efficient allocation the ratio of individuals' marginal utilities of consumption in any state are constant, and equal to the ratio of Pareto weights.*

Planner's problem

- Write this as a Lagrangean with a multiplier $\lambda_t(s^t)$ on every resource constraint

$$\mathcal{L} = \max_{c_t^i(s^t)} \sum_i \theta^i \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\left(c_t^i(s^t)\right) - \sum_{t=0}^{\infty} \sum_{s^t|s_0} \lambda_t(s^t) \left[\sum_i y^i(s_t) - \sum_i c_t^i(s^t)\right]$$

- First order conditions for $c_t^i(s^t)$ and $c_t^j(s^t)$ for two individuals i and j

$$\begin{aligned} \beta^t \pi(s^t) \theta^i u^{i\prime}\left(c_t^i(s^t)\right) &= \lambda_t(s^t) \\ \beta^t \pi(s^t) \theta^j u^{j\prime}\left(c_t^j(s^t)\right) &= \lambda_t(s^t) \end{aligned}$$

- Combining these

$$\frac{u^{i\prime}\left(c_t^i(s^t)\right)}{u^{j\prime}\left(c_t^j(s^t)\right)} = \frac{\theta^j}{\theta^i}$$

- Corollary - *If* $u^i = u$ *for all* i, *and* $\theta^i = 1$ *for all* i *(utilitarian), then* $c^i(s^t) = Y(s_t)/N$.

Planner's problem - Example 1

- Suppose that $u^i(c) = u^j(c) = \log(c)$ and two individuals

$$\frac{u^{1\prime}(c_t^1(s^t))}{u^{2\prime}(c_t^2(s^t))} = \frac{\theta^2}{\theta^1} \quad \rightarrow \quad \frac{c_t^2(s^t)}{c_t^1(s^t)} = \frac{\theta^2}{\theta^1} \quad \rightarrow \quad c_t^2(s^t) = \frac{\theta^2}{\theta^1} c_t^1(s^t)$$

- Resource constraint

$$c_t^1(s^t) + c_t^2(s^t) = y^1(s_t) + y^2(s_t) =: Y(s_t)$$

- Combined

$$c_t^1(s^t) + \frac{\theta_2}{\theta_1} c_t^1(s^t) = Y(s_t)$$

- *Efficient allocation* - Each individual consumes a constant share of the aggregate endowment

$$c_t^1(s^t) = \frac{\theta_1}{\theta_1 + \theta_2} Y(s_t)$$

$$c_t^2(s^t) = \frac{\theta_2}{\theta_1 + \theta_2} Y(s_t)$$

Planner's problem - Example 1

- *Efficient allocation* - Each individual consumes a constant share of the aggregate endowment

$$c_t^1(s^t) = c^1(s_t) = \frac{\theta_1}{\theta_1 + \theta_2} Y(s_t)$$

$$c_t^2(s^t) = c^2(s_t) = \frac{\theta_2}{\theta_1 + \theta_2} Y(s_t)$$

- The efficient allocation exhibits *perfect consumption insurance*

- Individuals consume a constant fraction of the *aggregate endowment* $Y(s_t)$ regardless of their idiosyncratic endowments

- Individuals are still exposed to *aggregate risk*, as consumption still depends on the aggregate state

- Consumption is independent of past consumption, states, etc. Only depends on $Y(s_t)$. *Can we show that this is true more generally?*

Planner's problem

- Optimality condition

$$\frac{u^{i\prime}(c_t^i(s^t))}{u^{j\prime}(c_t^j(s^t))} = \frac{\theta^j}{\theta^i} \qquad \rightarrow \qquad c_t^j(s^t) = u^{j\prime -1}\left(\frac{\theta^i}{\theta^j} u^{i\prime}(c_t^i(s^t))\right)$$

- More generally, one would need a computer to solve:

$$c_t^i(s^t) + \sum_{j \neq i} u^{j\prime -1}\left(\frac{\theta^i}{\theta^j} u^{i\prime}(c_t^i(s^t))\right) = Y(s_t)$$

- But inspecting this condition it is immediately clear that $c_t^i(s^t) = c^i(s_t)$
- Result - *In the efficient allocation each individual's consumption is a function only of the aggregate endowment $Y(s_t)$, and independent of the history of states and individual endowments in state $y^i(s_t)$.*
- Result - *As $y_t^i(s^t) = y^i(s_t)$ and s_t is Markov, then $Y(s_t) = \sum_i y = i(s_t)$ is Markov, so $c_t^i(s^t) = c^i(s_t)$ is Markov.*
- In the efficient allocation the planner divides aggregate output among individuals in 'the same way' every period. Individuals are *completely insured* against any idiosyncratic risk they would experience through fluctuations in $y_t^i(s_t)$. Literally *"all in this together"*.

Planner's problem - Example 2

- Suppose that $u^1(c) = \log c$, and $u^2(c) = c$ and two individuals

$$\frac{u^{1\prime}(c^1(s_t))}{u^{2\prime}(c^2(s_t))} = \frac{\theta^2}{\theta^1} \qquad \rightarrow \qquad \frac{1}{c^1(s_t)} = \frac{\theta^2}{\theta^1}$$

- Resource constraint

$$c^1(s_t) + c^2(s_t) = Y(s_t)$$

- Efficient allocation

$$c^1(s_t) = \frac{\theta_1}{\theta_2}$$

$$c^2(s_t) = Y(s_t) - \frac{\theta_1}{\theta_2}$$

- In the efficient allocation the planner 'insures' individual 1 (concave utility)
- Allows the consumption of individual 2 to fluctuate along with $Y(s_t)$

Planner's problem

- Can we recover something like an *Euler equation*?
- First order conditions for consumption to i in states s^t and s^{t+1}

$$\begin{aligned} \beta^t \pi(s^t)\theta^i u^{i\prime}(c_t^i(s^t)) &= \lambda_t(s^t) \\ \beta^{t+1}\pi(s^{t+1})\theta^i u^{i\prime}(c_{t+1}^i(s^{t+1})) &= \lambda_{t+1}(s^{t+1}) \end{aligned}$$

- Combining these Recall: $\pi(s_{t+1}|s^t) := \pi(s^{t+1})/\pi(s^t)$

$$\beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c_{t+1}^i(s^{t+1}))}{u^{i\prime}(c_t^i(s^t))} = \frac{\lambda_{t+1}(s_{t+1}, s^t)}{\lambda_t(s^t)}$$

- Result - *In the efficient allocation the ratio of individuals' marginal utilities of consumption across states are equal.* $\rightarrow$ *Agree on the prices of assets bought in s^t and payoff in any and all $(s_{t+1}|s^t)$!*
  - Why? The righthand-side is independent of i
- Suppose that s_{t+1} all have same probability, then the marginal utility of consumption will be *increasing* iff $\lambda_{t+1}(s_{t+1}, s^t) > \lambda_t(s^t)$. In the efficient allocation the rate of change in marginal utilities of consumption are the same for everyone, and determined by the tightness of the planner's resource constraint.
- Decentralization clue - These look like *shadow prices* for an asset that we haven't yet defined!

Planner's problem

- Using our previous result that $c_t^i(s^t) = c^i(s_t)$ only depends on s_t

$$\frac{\lambda_{t+1}(s_{t+1}, s^t)}{\lambda_t(s^t)} = \beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c^i(s_{t+1}))}{u^{i\prime}(c^i(s_t))}$$

- Since the righthand-side depends only on s_{t+1}, s_t, then so too must the lefthand-side

$$\frac{\lambda(s_{t+1})}{\lambda(s_t)} = \beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c^i(s_{t+1}))}{u^{i\prime}(c^i(s_t))}$$

- Re-arranging and summing over $s_{t+1}|s_t$

Price of a bond in RBC model: $Q_t = \mathbb{E}\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}\frac{1}{Q_t}\right]$

$$u^{i\prime}(c^i(s_t)) = \beta\sum_{s_{t+1}|s_t}\pi(s_{t+1}|s_t)\left[\frac{\lambda(s_t)}{\lambda(s_{t+1})}u^{i\prime}(c^i(s_{t+1}))\right]$$

$$1 = \mathbb{E}\left[\beta\frac{u^{i\prime}(c^i(s_{t+1}))}{u^{i\prime}(c^i(s_t))}\frac{1}{\Lambda(s_{t+1}|s_t)}\middle|s_t\right] \quad , \quad \Lambda(s_{t+1}|s_t) := \frac{\lambda(s_{t+1})}{\lambda(s_t)}$$

- These are well-defined *shadow prices* of a particular asset bought in s_t paying one unit in s_{t+1}
- These shadow prices, also only depend on s_t

Decentralized equilibrium

- We want to understand the answer to our standard two questions...
- Question - *Can the efficient allocation be decentralized?*
- Question - *What markets are necessary?*
- Thought experiment: Does there exist a kind of asset that individuals could trade, such that in the competitive equilibrium the efficient allocation is obtained?
- If the answer is *YES* then we would have a strong understanding about the conditions on asset markets that are required for efficiency (which we've argued is about *insurance*) in the economy.
- Gives a *benchmark* that we can start deviating from / recognize when another economy is equiv. to
- We will call this benchmark a *complete markets* economy ... for reasons that will become clear
- Anything else is an *incomplete markets* economy and we will study a particular type of incomplete markets economy next week.

"All happy families are alike, but every unhappy family is unhappy in its own way" - Tolstoy

Two steps

- Step 1
  - Consider an easy to compute equilibrium that lacks realism
  - *Date-0 trading equilibrium*
  - Show that the competitive equilibrium is efficient
- Step 2
  - Consider a harder to compute equilibrium that is more realistic
  - *Sequential trading equilibrium*
  - Show that the allocations of the two coincide

Arrow-Debreu securities

- Named after economist Kenneth Arrow and Gerard Debreu
- Consider the following security (or ‘contract’, or ‘claim’), which is only ever traded in period 0
- At date-0, pay a price $q_t^0(s^t)$
- After history s^t, the security pays out 1 unit
- The *date-0 budget constraint* of individual i is

$$\underbrace{\sum_{t,s^t} q_t^0(s^t) c_t^i(s^t)}_{\text{Buy claims to future consumption}} \quad \leq \quad \underbrace{\sum_{t,s^t} q_t^0(s^t) y^i(s_t)}_{\text{Sell claims to future income}}$$

Competitive equilibrium

A *Date-0 trading equilibrium* is an allocation $\{c_t^i(s^t)\}_{i,t,s^t}$, and Date-0 prices $\{q_t^0(s^t)\}_{t,s^t}$ *such that*

1. *Optimality* - For each individual i, taking the Date-0 prices $\{q_t^0(s^t)\}_{t,s^t}$ as given, the allocation $\{c_t^i(s^t)\}_{t,s^t}$ solves

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i(c_t^i(s^t))$$

subject to

$$\sum_{t,s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t,s^t} q_t^0(s^t) y^i(s_t)$$

2. *Market clearing* - In each state the resource constraint is satisfied

$$\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t) \qquad , \qquad \text{for all } t, s^t$$

Competitive equilibrium

- Lagrangean for i:

$$\mathcal{L}^i = \max_{c_t^i(s^t), a_{t+1}^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i(c_t^i(s^t)) + \mu^i \sum_{t,s^t} q_t^0(s^t) \left[y^i(s_t) - c_t^i(s^t) \right]$$

- First order conditions

$$c_t^i(s^t): \qquad 0 = \beta^t \pi(s^t) u^{i\prime}(c_t^i(s^t)) - \mu^i q_t^0(s^t)$$

- Combined for individuals i and j

$$\frac{u^{i\prime}(c_t^i(s^t))}{u^{j\prime}(c_t^j(s^t))} = \frac{\mu^i}{\mu^j}$$

Competitive equilibrium

- Lagrangean for i:

$$\mathcal{L}^i = \max_{c_t^i(s^t), a_{t+1}^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i(c_t^i(s^t)) + \mu^i \sum_{t,s^t} q_t^0(s^t) \Big[y^i(s_t) - c_t^i(s^t) \Big]$$

- First order conditions

$$c_t^i(s^t): \qquad 0 = \beta^t \pi(s^t) u^{i\prime}(c_t^i(s^t)) - \mu^i q_t^0(s^t)$$

$$c_t^i(s^t): \qquad 0 = \beta^t \theta^i \pi(s^t) u^{i\prime}(c_t^i(s^t)) - \lambda_t(s^t)$$

- Combined for individuals i and j

$$\frac{u^{i\prime}(c_t^i(s^t))}{u^{j\prime}(c_t^j(s^t))} = \frac{\mu^i}{\mu^j} \qquad , \qquad \frac{u^{i\prime}(c^i(s_t))}{u^{j\prime}(c^j(s_t))} = \frac{\theta^j}{\theta^i}$$

First welfare theorem

- Combined for individuals i and j

$$\frac{u^{i\prime}(c^i(s_t))}{u^{j\prime}(c^j(s_t))} = \frac{\mu^i}{\mu^j} \qquad , \qquad \frac{u^{i\prime}(c^i(s_t))}{u^{j\prime}(c^j(s_t))} = \frac{\theta^j}{\theta^i}$$

✓ First welfare theorem - The allocation associated with the Date-0 trading equilibrium delivers *an* efficient allocation

- Which efficient allocation does the competitive equilibrium pick out?
- The one in which $\theta^i = 1/\mu^i$
- Intuition - The competitive equilibrium reflects the choices of a social planner with Pareto weights that put the highest weight on the individuals with the *lowest* multipliers on their constraints. These are individuals with large endowments, for which the social cost of delivering utility is low.
- Criticism - Fine ... but maybe we don't want this type of equilibrium. Worth remembering that there are *other* efficient allocations out there!
- 2nd WT: Can choose ex-ante lump sum transfers such that CE attains *any* efficient allocation ex-post

Writing down a recursive budget constraint

- Relative prices - Let $q^t_\tau(s^\tau)$ denote the *Date-t* prices

$$q^t_\tau(s^\tau) := \frac{q^0_\tau(s^\tau)}{q^0_t(s^t)} \quad \left(= \beta^{\tau-t}\pi(s_\tau|s_t)\frac{u^{i\prime}(c^i(s_\tau))}{u^{i\prime}(c^i(s_t))}\right) \quad \therefore \text{ A function of only } s_\tau \text{ and } s_t$$

- Net wealth - Let $\gamma^i_t(s^t)$ denote the *net wealth* of individual i from history s^t onwards. Also only depends on s_t:

$$\gamma^i(s_t) := \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} q^t_\tau(s_\tau)\Big[c^i(s_\tau) - y^i(s_\tau)\Big] \qquad \text{Note:} \quad \sum_i \gamma^i(s_t) = 0$$

- This is the amount of wealth that individual i in state s^t must hold in order to honor all future claims
- At $t = 0$, the right hand side is just the budget constraint, so $\gamma^i(s_0) = 0$.
- Let's try to represent this *recursively*

$$\gamma^i(s_t) = c^i(s_t) - y^i(s_\tau) + \sum_{s_{t+1}|s_t} q^t_{t+1}(s_{t+1})\gamma^i(s_{t+1})$$

$$c^i(s_t) + \sum_{s_{t+1}|s_t} q^t_{t+1}(s_{t+1})\gamma^i(s_{t+1}) = y^i(s_\tau) + \gamma^i(s_t) \qquad \text{This looks like a budget constraint!}$$

Arrow securities

- Named *Arrow securities* named after economist Kenneth Arrow.
- Security: (1) Pay a price $Q_t(s_{t+1}, s^t)$ after history s^t to buy one Arrow security 'into' s_{t+1}
- (2) If individual i holds $a^i_{t+1}(s_{t+1}, s^t)$ securities into state s_{t+1}, then get $a^i_{t+1}(s_{t+1}, s^t)$ units of consumption goods if s_{t+1} occurs
- Suppose that *markets are complete* in that there is a complete set of such securities available for each s_{t+1} after any s^t. Pause to think about what this entails...
- The *budget constraint* of individual i after history s^t

$$c^i_t(s^t) + \underbrace{\sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a^i_{t+1}(s_{t+1}, s^t)}_{\text{Portfolio of securities}} \leq y^i(s_t) + a^i_t(s^t)$$

- *Market clearing* - All security positions must net out to zero (right?)

$$\sum_i a^i_{t+1}(s_{t+1}, s^t) = 0$$

$$c^i(s_t) + \sum q^t_{t+1}(s_{t+1})\gamma^i(s_{t+1}) = y^i(s_\tau) + \gamma^i(s_t)$$

Competitive equilibrium

A *sequential trading equilibrium* is an initial distribution of wealth $\{a_0^i(s_0)\}_i$, an allocation $\{c_t^i(s^t)\}_{i,t,s^t}$, and prices $\{Q_t(s_{t+1}, s^t)\}_{t,s^t}$

such that

1. For each individual i, taking prices $\{Q_t(s_{t+1}, s^t)\}_{t,s^t}$ as given, the allocation $\{c_t^i(s^t)\}_{t,s^t}$ solves

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i(c_t^i(s^t))$$

subject to

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) \leq y^i(s_t) + a_t^i(s^t)$$

2. In each state markets clear. For all t, s^t:

$$\underbrace{\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)}_{\text{Goods}} \quad , \quad \underbrace{\sum_i a_{t+1}^i(s_{t+1}, s^t) = 0}_{\text{Assets}}$$

Competitive equilibrium

- Lagrangean for i:

$$\mathcal{L}^i = \max_{c_t^i(s^t), a_{t+1}^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i(c_t^i(s^t))$$

$$+ \sum_{t=0}^{\infty} \sum_{s^t|s_0} \eta_t^i(s^t) \left[y^i(s_t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) \right]$$

- First order conditions

$$\begin{aligned} c_t^i(s^t): \qquad 0 &= \beta^t \pi(s^t) u^{i\prime}(c_t^i(s^t)) - \eta_t^i(s^t) \\ a_{t+1}^i(s_{t+1}, s^t): \qquad 0 &= \eta_t^i(s^t) Q_t(s_{t+1}, s^t) + \eta_{t+1}^i(s_{t+1}, s^t) \end{aligned}$$

- Combined

$$Q_t(s_{t+1}, s^t) = \frac{\eta_{t+1}^i(s_{t+1}, s^t)}{\eta_t^i(s^t)} = \beta \pi(s_{t+1}|s^t) \frac{u^{i\prime}(c_{t+1}^i(s_{t+1}, s^t))}{u^{i\prime}(c_t^i(s^t))}$$

Competitive equilibrium

- From the Date-0 trading equilibrium

$$\frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = \beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c^i(s_{t+1}))}{u^{i\prime}(c^i(s_t))}$$

- From the efficient allocation

$$\frac{\lambda(s_{t+1})}{\lambda(s_t)} = \beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c^i(s_{t+1}))}{u^{i\prime}(c^i(s_t))}$$

- Combined

$$Q_t(s_{t+1}, s^t) = \beta\pi(s_{t+1}|s_t)\frac{u^{i\prime}(c^i_{t+1}(s_{t+1}, s^t))}{u^{i\prime}(c^i_t(s^t))} = q^t_{t+1}(s_{t+1}, s^t)$$

- Result - The allocations coincide if $a^i(s_0) = 0$ for all i

Constructive proof

1. Solve the Date-0 trading equilibrium

3. Compute $Q_t(s_{t+1}, s^t) = q^0_{t+1}(s^{t+1})/q^0_t(s^t)$

*We then know that **if** the consumption allocation of both equilibria coincide, then individuals' first order conditions are satisfied*

4. Compute $a^i_{t+1}(s_{t+1}, s^t) = \gamma^i(s_t)$, and set $a^i_0(s_0) = 0$.

We then know that the consumption allocations of both equilibria coincide.

$$
\begin{aligned}
c^i_t(s^t) + \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a^i_{t+1}(s_{t+1}, s^t) &= y^i(s_t) + a^i_t(s^t) \\
c^i(s_t) + \sum_{s_{t+1}|s_t} q^t_{t+1}(s_{t+1}) \gamma^i(s_{t+1}) &= y^i(s_t) + \gamma^i(s_t)
\end{aligned}
$$

- Other results
- Asset holdings depend only on s_{t+1}: $a^i_{t+1}(s_{t+1}, s_t) = \gamma^i(s_{t+1})$

  $\rightarrow$ Individuals are always fully insured up to s^t, then buy insurance one period ahead
- Prices are Markov: $Q_t(s_{t+1}, s^t) = Q(s_{t+1}, s_t)$

Bellman

- As usual is seems a bit weird to write this down as a period-0 problem, can we Bellman-ize it?
- *State variables* - Asset position $a^i(S)$, state S
- *Prices* - Prices of Arrow securities $Q(S', S)$ Like P(S) in last lecture
- Let $V^i(a, S)$ be the present discounted value of utility to individual i with assets a when the state is S:

$$V^i(a,S) = \max_{\{a(S')\}_{S'}} u^i(c) + \beta \sum_{S'|S} \pi(S'|S) V^i\left(a(S'), S'\right)$$

Choosing a *portfolio* of a's into all S''s

subject to

$$c + \sum_{S'} Q(S,S')a(S') = y^i(S) + a(S)$$

- Euler equation

$$u'(c^i(S)) = \beta \mathbb{E}\left[u^{i\prime}(c^i(S'))\frac{1}{Q(S,S')}\middle| S\right]$$

Complete markets

Remark 1

- *Completeness* - Individuals can trade assets into *every* state *every* period
- *Insurance* - The competitive equilibrium provides *full insurance* against idiosyncratic risk
- *Pareto weights* - The C.E. is corresponds to Eff. allocation with $\theta^i = 1/\mu_i$. High weight on rich guys.
- Think of this as *complete risk-sharing*
- Note that no asset prices reflect an individual's risk. They only depend on the aggregate state.
- There is no *return premium* associated with bearing *idiosyncratic risk*

Remark 2

- Technically don't need assets that individually pay into each state
- If there are two states $s_t \in \{s^L, s^H\}$, we could have two assets, one which pays off in both states, and one that pays off only in state s^L.
- Technically, require the *matrix of payoffs* to have *full rank*.
- Then individuals can *reconstruct Arrow security payoffs* by holding portfolios of these assets

Questions

1. Can we 'complete' (verb) markets, when there is a not particularly rich structure of assets out there in the economy?

2. Is there a link between complete markets and the fact that we were comfortable working with a 'representative household' earlier on in the course?

Complet(ing) markets

- Consider an economy with N individuals with potentially different u^i and endowments
- Three states $s_t \in \{s_1, s_2, s_3\}$. Call these {recession, normal, boom}
- Endowments - Each individual has $y^i(s_t)$.
- Arrow securities - Payoffs

	$a(s_1)$	$a(s_2)$	$a(s_3)$
s_1	1	0	0
s_2	0	1	0
s_3	0	0	1

- Now consider an economy with no Arrow securities
- Only one physical asset in the economy, *a tree*
- Payoffs of the tree

	$Y(s)$
s_1	$1 = 2 - 1$
s_2	$2 = 2 + 0$
s_3	$3 = 2 + 1$

- *Can we create securities such that we can complete markets? i.e. Get rid of <u>all</u> idiosyncratic risk.*

Complet(ing) markets

- Consider the following two securities, which we call *options*
- Option 1, is a *call option* with a pay off $c(s) = \max\{Y(s) - 2, 0\}$
- Option 2, is a *put option* with a pay off $p(s) = \max\{2 - Y(s), 0\}$

	$Y(s)$	$c(s)$	$p(s)$
s_1	1	0	1
s_2	2	0	0
s_3	3	1	0

- Can we create our set of Arrow securities?

$$
\begin{aligned}
a(s_1) &= p(s) \\
a(s_2) &= Y(s) - 3\,c(s) - 2\,p(s) \\
a(s_3) &= c(s)
\end{aligned}
$$

- Result - Derivative contracts allow us to enrich the set of state-dependent payoffs possible from a limited set of physical assets. Efficient allocations can therefore be attained.

Implications for 'representative agent' macroeconomics

- We've spent a lot of time this quarter studying a macroeconomy with a *representative household*
  - Continuum of identical households, choose the same C_t, I_t, and have flow utility $U(C_t, N_t)$
  - Continuum of identical firms, choose the same K_t, N_t, and operates production technology $Y_t = Z_t F(K_t, N_t)$ where $Z_{t+1} \sim \pi_Z(Z'|Z_t)$
- With complete markets, we know that $c_t^i = \widetilde{\theta}^i C_t$. Intuitively we can split out the idiosyncratic and aggregates.
- Are the two linked?

Competitive equilibrium

Given initial conditions , , a *sequential trading equilibrium* is an initial distribution of wealth $\{a_0^i(s_0), \quad\}_i$ an allocation $\{c_t^i(s^t), \quad\}_{i,t,s^t}$, prices $\{Q_t(s_{t+1}, s^t), \quad , \quad\}_{t,s^t}$ *such that*

1. *Individual optimality*: Taking prices as given, allocation $\{c_t^i(s^t), \quad\}_{t,s^t}$ solves

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\left(c_t^i(s^t)\right)$$

subject to

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) \quad \leq \quad \underbrace{e^i(s_t)}_{e_{t+1}^i \sim \pi_e(e'|e_t^i)\ ,\ \mathbb{E}[e]=1} \quad + a_t^i(s^t)$$

Here I'm using $e_t^i(s_t)$ to denote individual endowments, and the average of them across people is always 1

3. *Market clearing*: In each state markets clear. For all t, s^t

$$\underbrace{\sum_i c_t^i(s^t) \leq \sum_i e^i(s_t)}_{\text{Goods}} \quad , \quad \underbrace{\sum_i a_{t+1}^i(s_{t+1}, s^t) = 0}_{\text{Arrow securities}} \quad ,$$

Competitive equilibrium

Given initial conditions , $k_0^i(s_0)$, a *sequential trading equilibrium* is an initial distribution of wealth $\{a_0^i(s_0), k_0^i(s_0)\}_i$ an allocation $\{c_t^i(s^t), k_{t+1}^i(s^t)\}_{i,t,s^t}$, prices $\{Q_t(s_{t+1}, s^t),$ $, R_t(s^t)\}_{t,s^t}$ *such that*

1. *Individual optimality*: Taking prices as given, allocation $\{c_t^i(s^t), k_{t+1}^i(s^t)\}_{t,s^t}$ solves

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\left(c_t^i(s^t)\right)$$

subject to

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) + k_{t+1}^i(s^t) \leq \underbrace{e_t^i(s_t)}_{e_{t+1}^i \sim \pi_e(e'|e_t^i),\ \mathbb{E}[e]=1} + a_t^i(s^t) + (1-\delta)k_t^i(s^t) + R_t(s^t)k_t^i(s^t)$$

3. *Market clearing*: In each state markets clear. For all t, s^t

$$\underbrace{\sum_i c_t^i(s^t) \leq \sum_i e^i(s_t)}_{\text{Goods}}, \quad \underbrace{\sum_i a_{t+1}^i(s_{t+1}, s^t) = 0}_{\text{Arrow securities}}, \quad \underbrace{\sum_i k_t^i(s^t) = K_t(s^t)}_{\text{Capital}},$$

Competitive equilibrium

Given initial conditions $e_0^i(s_0), Z_0(s_0), k_0^i(s_0)$, a *sequential trading equilibrium* is an initial distribution of wealth $\{a_0^i(s_0), k_0^i(s_0)\}_i$ an allocation $\{c_t^i(s^t), k_{t+1}^i(s^t)\}_{i,t,s^t}$, prices $\{Q_t(s_{t+1}, s^t), W_t(s^t), R_t(s^t)\}_{t,s^t}$ *such that*

1. *Individual optimality*: Taking prices as given, allocation $\{c_t^i(s^t), k_{t+1}^i(s^t)\}_{t,s^t}$ solves

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t) u^i\Big(c_t^i(s^t)\Big)$$

subject to

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) + k_{t+1}^i(s^t) \leq \underbrace{W_t(s^t) e_t^i(s_t)}_{e_{t+1}^i \sim \pi_e(e'|e_t^i),\ \mathbb{E}[e]=1} + a_t^i(s^t) + (1-\delta) k_t^i(s^t) + R_t(s^t) k_t^i(s^t)$$

2. *Firm optimality*: The allocation $K_t(s^t)$ and labor $N_t(s^t)$ solves

$$\max_{K_t(s^t), N_t(s^t)} \underbrace{Z_t(s^t)}_{Z_{t+1} \sim \pi_Z(Z'|Z_t),\ ,:\mathbb{E}[Z]=1} F\Big(K_t(s^t), N_t(s^t)\Big) - W_t(s^t) N_t(s^t) - R_t(s^t) K_t(s^t)$$

3. *Market clearing*: In each state markets clear. For all t, s^t

$$\underbrace{\sum_i c_t^i(s^t) + i_t^i(s^t) \leq \sum_i y^i(s_t)}_{\text{Goods}} \quad , \quad \underbrace{\sum_i a_{t+1}^i(s_{t+1}, s^t) = 0}_{\text{Arrow securities}} \quad , \quad \underbrace{\sum_i k_t^i(s^t) = K_t(s^t)}_{\text{Capital}} \quad , \quad \underbrace{\sum_i e_t^i(s_t) = N_t(s^t) = \overline{N}}_{\text{Labor}}$$

Competitive equilibrium

- *Observation 1*: With complete markets, the economy is efficient, therefore $c_t^i(s^t) = \widetilde{\theta}^i C_t(s^t)$
  - Note that now $c_t^i(\cdot)$ is a constant share of *aggregate consumption*, which potentially is history dependent.
  - Here $\widetilde{\theta}^i$ is a constant and some function of Pareto weights. Exactly the same algebra as before.

- *Observation 2*: Individuals have the same set of Euler equation for Arrow secutirities as before plus an additional one for capital:

$$\underbrace{1 = \mathbb{E}\left[\frac{\beta u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s_t))}\frac{1}{Q_t(s_{t+1}|s^t)}\right]}_{\text{Arrow securities}} \quad , \quad \underbrace{1 = \mathbb{E}\left[\frac{\beta u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s_t))}\left[R_{t+1}(s^{t+1}) + (1-\delta)\right]\right]}_{\text{Capital}}$$

- *Observation 3*: Capital is a redundant asset from the point of view of the individual.
  - With access to a full set of Arrow securities, any individual can replicate payoffs to holding capital. As long as the rental rate of capital satisfies the Euler equation above, then individuals are indifferent as to how much they hold of it.

Representative household

- Suppose that $u^i(c)$ is the same for all i, and CRRA: $u^i(c) = c^{1-\sigma}/(1-\sigma)$

1. Individual Euler equation for capital under consumption allocation: $c_t^i(s^t) = \widetilde{\theta}^i C_t(s^t)$

$$1 = \mathbb{E}\left[\beta\left(\frac{C_{t+1}(s^{t+1})}{C_t(s^t)}\right)^{-\sigma}\left[R_{t+1}(s^{t+1}) + (1-\delta)\right]\middle| s^t\right]$$

2. Add up individual budget constraints under market clearing: $\sum_i a_{t+1}^i(s_{t+1}|s^t) = 0$

$$C_t(s_t) + K_{t+1}(s^t) = W_t(s^t)\overline{N} + (1-\delta)K_t(s^t) + R_t(s^t)K_t(s^t)$$

3. Firm optimality conditions under labor market clearing: $N_t(s^t) = \overline{N}$

$$Z_t(s^t)F_K(K_t(s^t), \overline{N}) = R_t(s^t) \quad , \quad Z_t(s^t)F_N(K_t(s^t), \overline{N}) = W_t(s^t)$$

- *These are the full set of equilibrium conditions of an economy with a representative household with CRRA preferences $U(C,N) = C^{1-\sigma}/(1-\sigma)$ and $\overline{N}$ workers. In this economy the state variables are $S_t = (K_t, Z_t)$.*

Comments

- Looking back, this is important for our interpretation of the representative agent models we spent most of our time studying in this course

- It says that we can think of all of these models as ones with a lot of heterogeneity among households *and complete markets* humming away in the background
  - For example, we can have arbitrary idiosyncratic risk, but still all our asset pricing results from last week 'go through'
  - Makes sense! All the idiosyncratic risk is looked after with complete markets, while aggregate risk cannot be insured

- In this case the economy is efficient in the sense of idiosyncratic risk as we have studied this week *(full insurance)* and efficient in terms of the aggregate allocation as we studied with the neoclassical model

- Is this general? Yes, just end up with more complicated U
  - Can be extended to heterogeneity in u^i
  - Can be extended to include labor supply n^i and $u^i(c_i, n_i)$

- Next lecture: Consider one particular *incomplete markets* economy, where this will no longer hold.