Honors - Economic Analysis III

Lecture 6: Dynamic Programming I

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This lecture

- Motivate dynamic programming approach
  - Neoclassical model under uncertainty $\rightarrow$ RBC model
- Terminology
  - State variables
  - Value function
  - Policy function
  - Markov chain
- Next: (i) Methods, (ii) Labor supply, (iii) RBC model

Environment - Centralized / Deterministic

- *Time* - Discrete $t = 0, 1, 2 \ldots$
- *Agents* - Representative household with N workers
- *Goods* - One good can either be used for consumption or investment

$$c_t + i_t = y_t$$

- *Endowments* - Household owns the initial capital stock k_0
- *Preferences* - Utility of the household at date 0 is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad \beta \in (0,1)$$

- *Technology* - Constant returns to scale production technology $y_t = f(k_t)$. Capital depreciates at rate δ

$$k_{t+1} = (1-\delta)k_t + i_t \quad , \quad \delta \in [0,1]$$

Problem

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the series of constraints

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t \quad , \quad t = 0, 1, 2, \ldots$$

and initial conditions

$$k_0 > 0$$

Environment - Centralized / Stochastic

- *Time* - Discrete $t = 0, 1, 2 \ldots$
- *Agents* - Representative household with N workers
- *Goods* - One good can either be used for consumption or investment

$$c_t + i_t = y_t$$

- *Endowments* - Household owns the initial capital stock k_0
- *Preferences* - Utility of the household at date 0 is

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad , \quad \beta \in (0,1)$$

- *Technology* - Constant returns to scale production technology $y_t = A_t f(k_t)$, where given A_0, $\{A_t\}_{t=1}^{\infty}$ is given by *stochastic process*:

$$\log A_{t+1} = (1-\rho) \log \bar{A} + \rho \log A_t + \underbrace{\varepsilon_{t+1}}_{\textbf{`Shock'}} \quad , \quad \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon)$$

Capital depreciates at rate δ

Why?

Kydland and Prescott (1982)

- In the data we observe large, persistent, fluctuations in A_t

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} N_t^{1-\alpha} \\
\log Y_t &= \alpha \log K_t + (1-\alpha) \log N_t + \log A_t.
\end{aligned}
$$

- **Q**: Can 'shocks' to A_t account for business cycles in the NC model?

Why do we care?

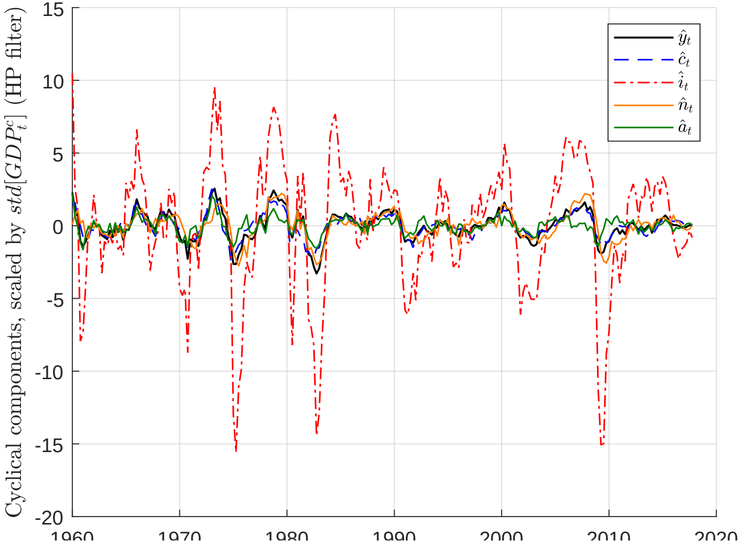
- As we learned in Lecture 4 ... NC model: (i) *Always in equilibrium*, (ii) *Welfare theorems hold*

- 1970s 'Keynesian' view: Recessions represent the economy being 'out of equilibrium' in some way. Government can intervene.

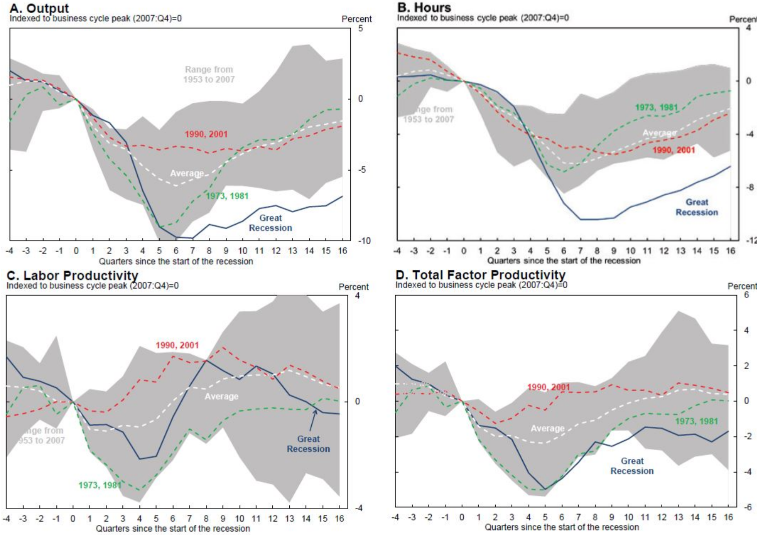
- Normative - If the answer is 'yes' then recessions are the economy's natural responses to exogenous shocks

- Positive - We have a quantitative theory of business cycles!

Business cycles - US data



Business cycles - US data



Fernald (2014) - Productivity and Potential Output Before, During, and After the Great Recession -

Business cycles - US data

		Volatility $std[\hat{x}_t]/std[\hat{y}_t]$	Covariance $corr(\hat{x}_t, \hat{y}_t)$	Persistence $corr(\hat{x}_t, \hat{x}_{t-1})$
Output (GDP)	$\hat{y}_t$	1	1	0.86
Consumption	$\hat{c}_t$	0.81	0.87	0.87
Investment	$\hat{i}_t$	4.52	0.90	0.83
Hours	$\hat{n}_t$	0.98	0.84	0.86

- Consumption - Less volatile than output
- Investment - Much volatile than output
- Hours - As volatile as output
- All series highly correlated with output

Problem in a not very useful format

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$

$$\max \ \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to the sequence of constraints Set $\delta = 1$

$$c_t + k_{t+1} \leq e^{a_t} f(k_t) \quad , \quad t = 0, 1, 2, \ldots$$

and initial conditions

$$k_0 > 0, \quad a_0$$

and stochastic process for $a_t = \log A_t - \log \bar{A}$

$$a_{t+1} = \rho a_t + \varepsilon_{t+1} \quad , \quad \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon)$$

Concept - State

- A *state* s_t is a finite vector of variables *pre-determined* at date-t
  - *Pre-determined - Values are realized before actions of saving / consumption occur*

- A *history* s^t is a t-period sequence of past states

$$s^t = \left(s_0, s_1, \ldots, s_{t-1}, s_t\right) = \left(s^{t-1}, s_t\right)$$

- Probability distribution $\pi(s^t) \in [0, 1]$ is defined over histories

$$\pi(s^t) = \pi(s_t, s^{t-1}) = \pi(s_t|s^{t-1})\pi(s^{t-1}) = \pi(s_t|s^{t-1})\pi(s_{t-1}|s^{t-2})\ldots\pi(s_1|s^0)\pi(s_0)$$

- *What is in the state vector?*
  - Good question!
  - For now we remain agnostic.
  - Use s^t as a way for accounting in the household problem
  - *Dynamic programming* approach will answer this question precisely!

Concept - State

- A *state* s_t is a finite vector of variables *pre-determined* at date-t

- A *history* s^t is a t-period sequence of past states

$$s^t = \Big(s_0, s_1, \ldots, s_{t-1}, s_t\Big) = \Big(s^{t-1}, s_t\Big)$$

- Probability distribution $\pi(s^t) \in [0,1]$ is defined over histories

$$\pi(s^t) = \pi(s_t, s^{t-1}) = \pi(s_t|s^{t-1})\pi(s^{t-1}) = \pi(s_t|s^{t-1})\pi(s_{t-1}|s^{t-2})\ldots\pi(s_1|s^0)$$

- *Stationarity*
  - We will study models where $\pi(s_{t+1}|s^t)$ depends only on s_t
  - If $s_t =$ 'Low', then no matter what date t is we have the same

$$\pi(s_{t+1} = \text{`}High'|s_t = \text{`}Low')$$

  - The state is a *Markov process* - *The probability distribution over states tomorrow, depends only on the state today*

Concept - State

- A *state* s_t is a finite vector of variables *pre-determined* at date-t

- A *history* s^t is a t-period sequence of past states

$$s^t = \Big(s_0, s_1, \ldots, s_{t-1}, s_t\Big) = \Big(s^{t-1}, s_t\Big)$$

- Probability distribution $\pi(s^t) \in [0,1]$ is defined over histories

$$\pi(s^t) = \pi(s_t, s^{t-1}) = \pi(s_t|s^{t-1})\pi(s^{t-1}) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_1|s_0)$$

- *Stationarity*
    - We will study models where $\pi(s_{t+1}|s_t)$ does not depend on t
    - If $s_t = \text{`}Low'$, then no matter what date t is we have the same

$$\pi(s_{t+1} = \text{`}High'|s_t = \text{`}Low')$$

    - The state is a *Markov process* - *The prob. distribution over states tomorrow, depends only on the state today*

Concept - State - Example

- States

$$a_t \in \{a^L, a^H\}$$

- Stochastic process - Consider a simple first-order *Markov process*

$$\begin{aligned} \pi_a(a^L|a^L) &= \pi_a(a^H|a^H) = 0.90 \quad , \quad \text{and} \\ \pi_a(a^L|a^H) &= \pi_a(a^H|a^L) = 0.10 \end{aligned}$$

- Initial condition

$$a_0 = a^H$$

- Probability distribution over states $\pi_a(a^t|a_0)$:
  - What is $\pi_a(a^t|a_0)$, when $a^t = (a^H, a^H, a^H)$?

$$\pi_a(a^t) = 0.90 \times 0.90 = 0.81$$

  - What is $\pi_a(a^t|a_0)$, when $a^t = (a^H, a^L, a^H)$?

$$\pi_a(a^t) = 0.10 \times 0.90 = 0.09$$

Problem in a not very useful format

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to the series of constraints

$$c_t + k_{t+1} \leq e^{a_t} f(k_t) \quad , \quad t = 0, 1, 2, \ldots$$

and initial conditions

$$k_0 > 0, \quad a_0$$

and stochastic process for a_t

$$a_{t+1} = \rho a_t + \varepsilon_{t+1} \quad , \quad \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon)$$

State-contingent Problem

Chooses state-contingent sequences $\{c_t(s^t), k_{t+1}(s^t)\}_{t=0,\forall s^t}^{\infty}$

$$\max_{\{c_t(s^t),k_{t+1}(s^t)\}} \left[\sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) u(c_t(s^t)) \right]$$

subject to the series of state-contingent constraints

$$c_t(s^t) + k_{t+1}(s^t) \leq e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad t = 0, 1, 2, \dots, \quad \forall s^t$$

and initial conditions

$$k_0(s_0) > 0 \quad , a_0(s_0)$$

and stochastic process for a_t

$$a_{t+1}(s^{t+1}) = \rho a_t(s^t) + \varepsilon_{t+1} \quad , \quad \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon)$$

Problem - Lagrangean

- Constrained optimization problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) u(c_t(s^t)) \tag{1}$$

$$+ \sum_{t=0}^{\infty} \sum_{s^t|s_0} \lambda_t(s^t) \left[e^{a_t(s^t)} f(k_t(s^t)) - c_t(s^t) - k_{t+1}(s^t) \right] \tag{2}$$

- First order necessary conditions

$$\begin{aligned} c_t(s^t): \quad \lambda_t(s^t) &= \pi(s^t|s_0)\beta^t u'(c_t(s^t)) \\ k_{t+1}(s^t): \quad \lambda_t(s^t) &= \sum_{s_{t+1}|s^0} \lambda_{t+1}(s_{t+1}, s^t) f'(k_{t+1}(s^t)) \end{aligned}$$

- Combining conditions

$$\pi(s^t|s_0)u'(c_t(s^t)) = \beta \sum_{s_{t+1}|s^0} \pi(s^{t+1}|s_0) u'(c_{t+1}(s^{t+1})) f'(k_{t+1}(s^t))$$

Problem - Lagrangean

- Combining conditions

$$\pi(s^t|s_0)u'(c_t(s^t)) = \beta \sum_{s^{t+1}|s_0} \pi(s_{t+1}, s^t|s_0)u'(c_{t+1}(s^{t+1}))f'(k_{t+1}(s^t))$$

- Conditional probabilities

$$\pi(s_{t+1}, s^t|s_0) = \pi(s_{t+1}|s^t)\pi(s^t|s_0)$$

- Using this

$$u'(c_t(s^t)) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t)u'(c_{t+1}(s^{t+1}))f'(k_{t+1}(s^t))$$

- More simply

$$u'(c_t(s^t)) = \beta \mathbb{E}\left[u'(c_{t+1}(s^{t+1}))f'(k_{t+1}(s^t))\middle| s^t\right]$$

Equilibrium conditions

- For all $t = 0, 1, 2, \ldots,$ and for all $s^t | s_0$

1. Euler equation

$$u'(c_t(s^t)) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) u'(c_{t+1}(s^{t+1})) f'(k_{t+1}(s^t))$$

2. Resource constraint

$$c_t(s^t) + k_{t+1}(s^t) \leq e^{a_t(s^t)} f(k_t(s^t))$$

3. Transversality condition

$$\lim_{T \to \infty} \beta^T \pi(s^T | s_0) u'(c_T(s^T)) k_{T+1}(s^T) = 0$$

Equilibrium conditions

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Equilibrium conditions

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1. Euler equation

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$$c_t(s^t) + k_{t+1}(s^t) \leq e^{a_t(s^t)} f(k_t(s^t))$$

3. Transversality condition

$$\lim_{T \to \infty} \beta^T \mathbb{E}\left[u'(c_T) k_{T+1} \middle| s_0 \right] = 0$$

- Still lots of questions!
  - State variables: What is in s_t?
  - How do decisions depend on s^t and t? Is it $c_t(s^t)$ or $c_t(s_t)$ or $c(s_t)$?
  - How to solve for $c_t(s^t)$, $k_{t+1}(s^t)$?

Value function

- *The value function* $V_t(s^t)$ *is the expected present discounted value of the household's utility under the optimal policy for consumption and capital, where this value is computed after the realization of the history* s^t, *but before the choice of* $c_t(s^t)$ *and* $k_{t+1}(s^t)$
- Period 0, state s_0 value function:

$$V_0(s_0) := \max_{\{c_t(s^t),k_{t+1}(s^t)\}} \left[\sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) u(c_t(s^t)) \right]$$

subject to the series of state-contingent constraints

$$c_t(s^t) + k_{t+1}(s^t) \quad = \quad e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 0, \forall s^t|s^0$$

Value function

- Period 0, state s_0 value function: Split up into $t = 0$ and $t \geq 1$

$$V_0(s_0) = \max_{c_0(s_0), k_1(s_0)} u(c_0(s_0))$$
$$+ \left\{ \max_{\{c_t(s^t), k_{t+1}(s^t)\}} \left[\sum_{t=1}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) u(c_t(s^t)) \right] \right\}$$

subject to the series of state-contingent constraints

$$\begin{aligned} c_0(s_0) + k_1(s_0) &= e^{a_0(s_0)} f(k_0(s_0)) \\ c_t(s^t) + k_{t+1}(s^t) &= e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 1, \forall s^t|s_1, s_0 \end{aligned}$$

Value function

- Period 0, state s_0 value function: $\pi(s^t|s_0) = \pi(s^t|s_1)\pi(s_1|s_0)$

$$
\begin{aligned}
V_0(s_0) = & \max_{c_0(s_0),k_1(s_0)} u(c_0(s_0)) \\
& + \left\{ \max_{\{c_t(s^t),k_{t+1}(s^t)\}} \left[\sum_{t=1}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_1)\pi(s_1|s_0) u(c_t(s^t)) \right] \right\}
\end{aligned}
$$

subject to the series of state-contingent constraints

$$
\begin{aligned}
c_0(s_0) + k_1(s_0) &= e^{a_0(s_0)} f(k_0(s_0)) \\
c_t(s^t) + k_{t+1}(s^t) &= e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 1, \forall s^t|s_1, s_0
\end{aligned}
$$

Value function

- Period 0, state s_0 value function: $\sum_{s^t|s_0} = \sum_{s^t|s_1} \sum_{s_1|s_0}$

$$V_0(s_0) = \max_{c_0(s_0),k_1(s_0)} u(c_0(s_0))$$
$$+ \sum_{s_1|s_0} \pi(s_1|s_0) \left\{ \max_{\{c_t(s^t),k_{t+1}(s^t)\}} \left[\sum_{t=1}^{\infty} \sum_{s^t|s_1} \beta^t \pi(s^t|s_1) u(c_t(s^t)) \right] \right\}$$

subject to the series of state-contingent constraints

$$c_0(s_0) + k_1(s_0) = e^{a_0(s_0)} f(k_0(s_0))$$

$$c_t(s^t) + k_{t+1}(s^t) = e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 1, \forall s^t | s_1, s_0$$

Value function

- Period 0, state s_0 value function: $\sum_{t=1}^{\infty} \beta^t = \beta \sum_{t=0}^{\infty} \beta^t$

$$
\begin{aligned}
V_0(s_0) = & \max_{c_0(s_0), k_1(s_0)} u(c_0(s_0)) \\
& + \beta \sum_{s_1|s_0} \pi(s_1|s_0) \left\{ \max_{\{c_t(s^t), k_{t+1}(s^t)\}} \left[\sum_{t=0}^{\infty} \sum_{s^t|s_1} \beta^t \pi(s^t|s_1) u(c_t(s^t)) \right] \right\}
\end{aligned}
$$

subject to the series of state-contingent constraints

$$
\begin{aligned}
c_0(s_0) + k_1(s_0) &= e^{a_0(s_0)} f(k_0(s_0)) \\
c_t(s^t) + k_{t+1}(s^t) &= e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 1, \forall s^t|s_1, s_0
\end{aligned}
$$

Value function

- Period 0, state s_0 value function:

$$
\begin{aligned}
V_0(s_0) = & \max_{c_0(s_0),k_1(s_0)} u(c_0(s_0)) \\
& + \beta \sum_{s_1|s_0} \pi(s_1|s_0) \left\{ \max_{\{c_t(s^t),k_{t+1}(s^t)\}} \left[\sum_{s^t|s_1} \sum_{t=0}^{\infty} \beta^t \pi(s^t|s_1) u(c_t(s^t)) \right] \right\}
\end{aligned}
$$

subject to the series of state-contingent constraints

$$
\begin{aligned}
c_0(s_0) + k_1(s_0) & = e^{a_0(s_0)} f(k_0(s_0)) \\
c_t(s^t) + k_{t+1}(s^t) & = e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 1, \forall s^t|s_1
\end{aligned}
$$

The terms in red are exactly $V_1(s_1)$.

(Go back and check the definition of $V_0(s_0)$).

Value function

- Period 0, state s_0 value function:

$$V_0(s_0) = \max_{c_0(s_0), k_1(s_0)} u(c_0(s_0)) + \beta \sum_{s_1|s_0} \pi(s_1|s_0) V_1(s_1)$$

subject to the constraint

$$c_0(s_0) + k_1(s_0) = e^{a_0(s_0)} f(k_0(s_0))$$

- Now we can determine what should be included in s
- We need to know everything such that we can (i) solve the household's decision given s_0, (ii) fully determine s_1 so that we can solve their problem again tomorrow.

Value function

- Suppose that we conjecture that the state is $s_t = (a_t, k_t)$

- Let's try

$$V_0(a_0, k_0) = \max_{c_0, k_1} u(c_0) + \beta \sum_{a_1, k_1 \mid a_0, k_0} \pi(a_1, k_1 \mid a_0, k_0) V_1(a_1, k_1)$$

subject to the constraint

$$c_0 + k_1 \quad = \quad e^{a_0} f(k_0)$$

- The k_1 part of s_1 is determined by the choice of capital today $k_1(a_0, k_0)$
- The a_1 part of s_1 is drawn from the stochastic process for a

$$a' \sim \pi_a(a'|a) \qquad \text{i.e.} \quad Prob[a_1|a_0] = \pi(a_1|a_0)$$

- Therefore the probability of (a_1, k_1) given (a_0, k_0) is

$$\pi(a_1, k_1|a_0, k_0) = \mathbf{1}[k' = k_1(a_0, k_0)] \times \pi_a(a'|a)$$

Value function

- Conjecture that $s_t = (a_t, k_t)$

$$V_0(a_0, k_0) = \max_{c_0, k_1} u(c_0) + \beta \sum_{a_1 | a_0} \pi_a(a_1 | a_0) V_1(a_1, k_1)$$

subject to the constraint

$$c_0 + k_1 \quad = \quad e^{a_0} f(k_0)$$

Value function

- Conjecture that $s_t = (a_t, k_t)$

$$V_0(a_0, k_0) = \max_{c_0, k_1} u(c_0) + \beta \sum_{a_1|a_0} \pi_a(a_1|a_0) V_1(a_1, k_1)$$

subject to the constraint

$$c_0 + k_1 = e^{a_0} f(k_0)$$

- In our example, had the following stochastic process:

$$\begin{aligned} a_1 &= \rho a_0 + \varepsilon_1 \quad , \quad \varepsilon_1 \sim N(0, \sigma_\varepsilon) \\ \Longrightarrow \quad \pi_a(a_1|a_0) &= Prob[a_1|a_0] = \phi\left(\frac{a_1 - \rho a_0}{\sigma_\varepsilon}\right) \end{aligned}$$

where $\phi(\cdot)$ is the probability under the standard normal distribution

Value function

- Note that time t plays no role in the value function

- All of the decisions depend only on the state, not the date t that decisions happen
- So we can drop time subscripts on $V_t(a_t, k_t)$ and on variables

$$V(a,k) = \max_{c,k'} \; u(c) + \beta \sum_{a'|a} \pi_a(a'|a) V(a',k')$$

subject to the constraint

$$c + k \quad = \quad e^a f(k)$$

Two problems that deliver same solution

1. Bellman equation

$$V(a,k) = \max_{c,k'} u(c) + \beta \sum_{a'|a} \pi_a(a'|a) V(a',k')$$

subject to the constraint

$$c + k = e^a f(k) \quad \rightarrow \quad \underbrace{c(a,k)\ ,\ k'(a,k)}_{\text{Policy functions}}$$

2. Sequence problem

$$V_0(s_0) := \max_{\{c_t(s^t), k_{t+1}(s^t)\}} \left[\sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) u(c_t(s^t)) \right]$$

subject to the series of state-contingent constraints

$$c_t(s^t) + k_{t+1}(s^t) = e^{a_t(s^t)} f(k_t(s^t)) \quad , \quad \forall t \geq 0, \forall s^t|s_0$$

Principle of Optimality

- Result - Take a plan $c_t(s^t)$ that solves the sequence problem. Then this sequence must also satisfy the Bellman equation.

  > *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision"* - Bellman (1957), "Dynamic Programming"

  > ... if given the state today, you are to make a particular decision, then tomorrow—if the state is the same—you should make the same decision

- Further result - This is a *necessary* condition. *"If [**plan**] solves sequence, then [**plan**] solves the Bellman"*. Under additional, easy to verify conditions, the Bellman equation is necessary *and* sufficient

- Analogy - Like sub-game perfection, with an infinite horizon!

- Analogy - Makes *perfectly clear* the notion of a 'sunk cost fallacy'. Past actions only affect decisions today in so far as they are coded into the current period state variables s_t.

Principle of Optimality

- Today, I solve a problem that determines how the state evolves

  Given (a_t, k_t) I choose c_t, which determines k_{t+1}

- Tomorrow, after the resolution of some uncertainty, I am faced with the exact same problem

  Given (a_{t+1}, k_{t+1}) I choose c_{t+1}, which determines k_{t+2}

- The *policy function* (optimal policy) gives the optimal decision in state s

  $c(a, k) \quad k'(a, k)$

- The *value function* gives the PDV of utility *under the optimal policy*

$$V(a, k) = \sum_{t=0}^{\infty} \sum_{(a_t, k_t)|(a,k)} \beta^t \pi(a_t, k_t) u(c(a_t, k_t))$$

- Result - The sequence $c_t(s^t)$ is generated by a policy function $c(s)$ (!!!)

Example 1 - A 'Cake-eating' problem

- Sequence problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$W_{t+1} = R(W_t - c_t) \quad \forall t \quad , \quad W_0 > 0 \quad , \quad W_{t+1} \geq 0$$

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- Bellman equation

$$V(W) = \max_c u(c) + \beta V(W')$$

subject to

$$W' = R(W - c) \quad , \quad W' \geq 0$$

Example 2 - A job search problem

- An individual begins period t with an offer of a job at wage w
- She can accept the job
  - In which case she works forever at wage $w_t = w$
- She can reject the job
  - In which case she receives unemployment payment b
    and at beginning of $t+1$ she draws a new job offer $w' \sim F(w)$ where $w' \in [0, \infty)$
- When working her consumption is $c_t = w_t$, when unemployed $c_t = b$
- Let $V(w)$ be the expected PDV of lifetime utility under the optimal policy

$$V(w) := \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right].$$

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- Result $\quad V(w)$ satisfies the Bellman equation

$$V(w) = \max\left\{ \quad \frac{u(w)}{1-\beta} \quad , \quad u(b) + \int_0^{\infty} V(w')dF(w')\right\}$$

General problem

1. Bellman equation

$$V(x,z) = \max_{x'} \; F(x,x',z) + \beta \mathbb{E}\left[V(x',z') \middle| z\right]$$

subject to the constraint

$$x' \in \Gamma(x,z)$$

2. Sequence problem

$$V(x_0,z_0) := \max_{\{x_{t+1}\}_{t=0}^{\infty}} \; \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t F\left(x_t, z_t, x_{t+1}\right)\right]$$

subject to the sequence of state-contingent constraints

$$x_{t+1} \in \Gamma(x_t, z_t)$$

General problem

1. Bellman equation

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subject to the constraint

$$x' \in \Gamma(x,z)$$

- Questions
  - If we have V, how do we find the policy $x'(x,z)$?
  - How do we find V?
  - How do we handle *competitive equilibrium* problems?
    - Lucas (1972), Mehra Prescott (1982)

END