Honors - Economic Analysis III

Lecture 1: Solow model

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This lecture

- Facts of economic growth
- Solow model of economic growth
- Steady state, comparative statics, transition dynamics
- *Questions*
  1. *Positive* - Can the model replicate the facts?
  2. *Normative* - Can welfare be lower / higher under different policies?

Growth facts

1. Share of income to capital and labor are approximately constant

$$\frac{\text{Labor payments}}{GDP} \approx 0.65 \quad , \quad \frac{\text{Capital payments}}{GDP} \approx 0.35$$

2. Capital-Output ratio is approximately constant

$$\frac{\text{Capital}}{\text{GDP}} \approx 3$$

3. Capital per worker and output per worker grow at constant rates

$$\frac{K_{t+1}/N_{t+1}}{K_t/N_t} \approx \gamma_k \quad , \quad \frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} \approx \gamma_n$$

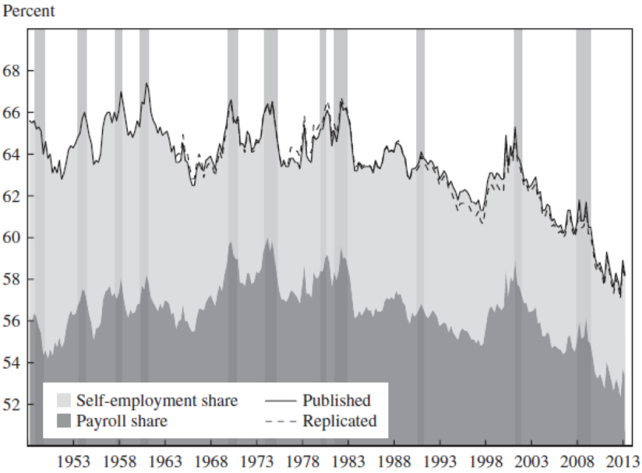
4. Capital grows at a constant rate

$$\frac{K_{t+1}}{K_t} \approx \gamma_K$$

1. Labor share

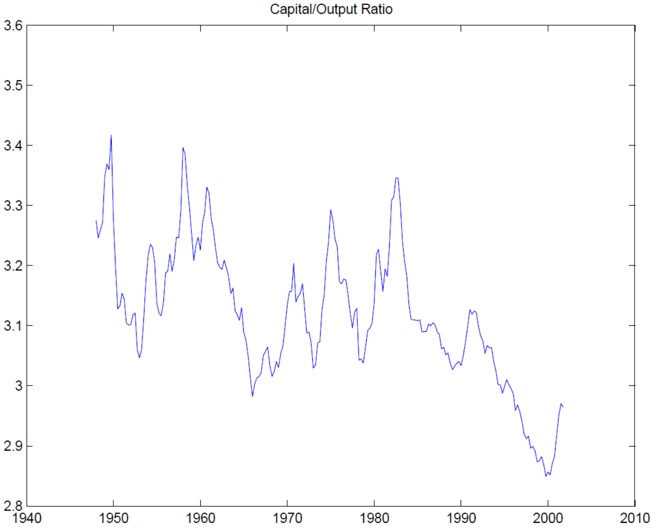
Labor share was previously constant but has recently fallen. Lots of research on this currently!

Figure 1. Labor Share, Payroll Share, and Replicated Labor Share in U.S. Nonfarm Business Sector, 1948-2013



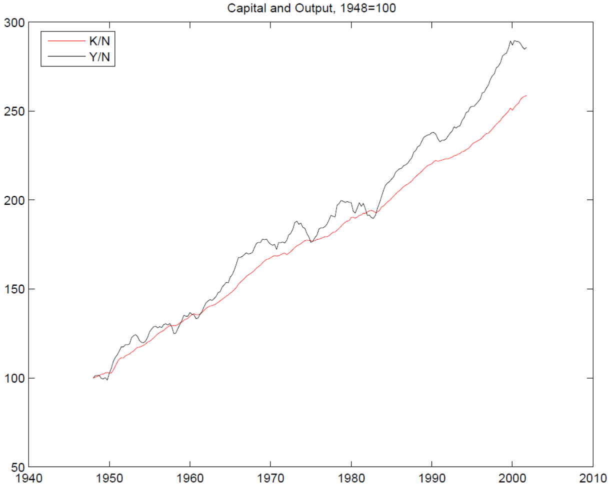
2. Capital / Output ratio

Capital/Output and the growth rates of Capital/Worker and Output/Worker are approximately constant.



3. Capital- and Output- per worker

Capital/Output and the growth rates of Capital/Worker and Output/Worker are approximately constant.



Solow model - Environment

- *Technology*
  - Firms operate CRS technology

$$Y_t = F(K_t, N_t) = K_t^{\alpha} \left(A_t N_t\right)^{1-\alpha}$$

- *Household preferences and behavior*
  - Households have a preference only for consumption

$$U(C_t, N_t) = C_t$$

  - Households receive income $Y_t = W_t N_t + R_t K_t + \Pi_t$, they invest

$$C_t = (1 - \xi) Y_t \quad , \quad I_t = \xi Y_t \quad , \quad K_{t+1} = (1 - \delta) K_t + I_t$$

- *Resources*
  - Goods: $Y_t = C_t + I_t$.

Competitive factor pricing - W_t

- Firms are *competitive*
- Profit maximization problem taking prices as given

$$\Pi_t = \max_{K_t, N_t} \; P_t \underbrace{F\left(K_t, A_t N_t, \right)}_{Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}} - R_t K_t - W_t N_t$$

- First order condition for N_t

$$N_t : \qquad 0 = P_t \times \left\{(1-\alpha) K_t^{\alpha} \left(A_t N_t\right)^{-\alpha} A_t\right\} - W_t$$

- *Result* - *Under constant returns to scale, competitive pricing of inputs implies that factor shares equal output elasticities*

$$\frac{W_t N_t}{P_t Y_t} = 1 - \alpha \quad , \quad W_t = 1 - \alpha \times \frac{P_t Y_t}{N_t}$$

Do the same for K_t. Satisfies first pair of growth facts. Verify $Y_t = R_t K_t + W_t N_t + \Pi_t$, with $\Pi_t = 0$. Given quantities we can read off prices.

Competitive factor pricing - W_t

- Firms are *competitive*
- Profit maximization problem taking prices as given

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Do the same for K_t. Satisfies first pair of growth facts. Verify $Y_t = R_t K_t + W_t N_t + \Pi_t$, with $\Pi_t = 0$. Given quantities we can read off prices.

Competitive factor pricing - R_t

- Firms are *competitive*
- Profit maximization problem taking prices as given

$$\Pi_t = \max_{K_t, N_t} \; P_t \underbrace{F\left(K_t, A_t N_t,\right)}_{Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}} - R_t K_t - W_t N_t$$

- First order condition for K_t

$$K_t : \qquad 0 = P_t \times \left\{ \alpha K_t^{\alpha-1} \left(A_t N_t\right)^{1-\alpha} A_t \right\} - R_t$$

- *Result* - *Under constant returns to scale, competitive pricing of inputs implies that factor shares equal output elasticities*

$$\frac{R_t K_t}{P_t Y_t} = \alpha \quad , \quad R_t = \alpha \times \frac{P_t Y_t}{K_t}$$

Do the same for K_t. Satisfies first pair of growth facts. Verify $Y_t = R_t K_t + W_t N_t + \Pi_t$, with $\Pi_t = 0$. Given quantities we can read off prices.

Determining quantities

- *Initial conditions* - $\{A_0, N_0, K_0\}$
- *Exogenous variables* - $\{A_{t+1}, N_{t+1}\}_{t=0}^{\infty}$ - Given by:

$$A_{t+1} = (1+\gamma_A)A_t \quad , \quad N_{t+1} = (1+\gamma_N)N_t.$$

- *Endogenous variables* - $\{K_{t+1}, Y_t, C_t, I_t\}_{t=0}^{\infty}$ - Determined by model:

$$\begin{aligned} Y_t &= F(K_t, A_t N_t) && (1) \\ K_{t+1} &= (1-\delta)K_t + I_t && (2) \\ Y_t &= C_t + I_t && (3) \\ C_t &= (1-\xi)Y_t && (4) \end{aligned}$$

- Each t, given $\{A_t, N_t, K_t\}$. We have 4 unknowns $\{K_{t+1}, Y_t, C_t, I_t\}$.
- These can be determined by the 4 equations
- The variable K_{t+1} evolves *endogenously* $\Rightarrow$ $K_{t+1} = G(K_t, A_t, N_t)$
- Can just focus on this important *endogenous state variable*

Determining quantities

Always count equations and unknowns!

- We can combine these to deliver the following set of equilibrium conditions at date t and $t+1$:

$$
\begin{aligned}
U_C\left(C_t, N_t\right) &= \beta U_C\left(C_{t+1}, N_{t+1}\right)\left[\left(1-\delta\right)+R_{t+1}\right] \\
C_t + K_{t+1} &= Y_t + \left(1-\delta\right) K_t \\
\frac{U_N\left(C_t, N_t\right)}{U_C\left(C_t, N_t\right)} &= W_t \\
\frac{U_N\left(C_{t+1}, N_{t+1}\right)}{U_C\left(C_t, N_t\right)} &= W_{t+1} \\
Y_t &= \mathcal{A}_t\left(K_t^{1-\gamma} N_t^{\gamma}\right)^{\alpha} \\
Y_{t+1} &= \mathcal{A}_{t+1}\left(K_{t+1}^{1-\gamma} N_{t+1}^{\gamma}\right)^{\alpha} \\
W_t N_t &= \mathcal{M}_t \alpha\gamma Y_t \\
W_{t+1} N_{t+1} &= \mathcal{M}_{t+1} \alpha\gamma Y_{t+1} \\
R_{t+1} K_{t+1} &= \alpha\left(1-\gamma\right) Y_{t+1}
\end{aligned}
$$

along with the transversality condition: $\lim_{T\to\infty} \beta^T U_C\left(C_t, N_t\right) = 0$

- Suppose that the path for $\left\{\mathcal{M}_t, \mathcal{A}_t\right\}_{t=0}^{\infty}$ is known. Given some initial capital K_0, there is a unique saddle path for variables such that the transversality condition holds. This is pinned down by C_0. Given $\left\{K_0, C_0\right\}$, then the above system gives
  - 9 equations. 9 unknowns: $\left\{N_0, Y_0, W_0, N_1, Y_1, W_1, R_1, K_1, C_1\right\}$
- In steady state:

Per-worker

- Production function

$$Y_t = F(K_t, A_t N_t) = A_t N_t F\left(\frac{K_t}{A_t N_t}, 1\right) = A_t N_t f\left(\frac{K_t}{A_t N_t}\right)$$

- Let $x_t = X_t / A_t N_t$ be X_t *per efficiency unit of labor*

$$y_t = f(k_t) = k_t^\alpha \quad , \quad \frac{Y_{t+1}}{Y_t} = \frac{A_{t+1} N_{t+1} y_{t+1}}{A_t N_t y_t} = \gamma_A \gamma_N \frac{y_{t+1}}{y_t}$$

- Given k_0 we have four equations in four unknowns $\{y_t, c_t, i_t, k_{t+1}\}$

$$\begin{aligned} y_t &= f(k_t) \\ k_{t+1}(1+\gamma_A)(1+\gamma_N) &= (1-\delta)k_t + i_t \\ y_t &= c_t + i_t \\ c_t &= (1-\xi)y_t \end{aligned}$$

- *Want:* Instead of $K_{t+1} = G(K_t, A_t, N_t)$, want $k_{t+1} = g(k_t)$.

Solution

- Using this we have four equations in four unknowns $\{k_{t+1}, y_t, c_t, i_t\}$

$$\begin{aligned} y_t &= f(k_t) \\ k_{t+1}(1+\gamma_A)(1+\gamma_N) &= (1-\delta)k_t + i_t \\ y_t &= c_t + i_t \\ c_t &= (1-\xi)y_t \end{aligned}$$

- Solving these

$$k_{t+1} = \frac{1-\delta}{(1+\gamma_A)(1+\gamma_N)} k_t + \frac{\xi}{(1+\gamma_A)(1+\gamma_N)} f(k_t) = g(k_t) \quad (*)$$

- Have we solved the model? Let's check ...

- Given K_0, A_0, N_0 we know $k_0 = K_0/A_0N_0$.
- From $(*)$ we then know $k_1, k_2, \ldots$
- We also know $\{A_t, N_t\}_{t=1}^{\infty}$, so we can get $\{K_t, N_t\}$, which gives Y_t, C_t, I_t

Steady-state

- What are the properties of the steady-state of the model?
- Should be able to express all endogenous variables in terms of only *parameters*
- Do the comparative statics w/r/t parameters make sense?
    - Higher or lower savings rate ξ?
    - Higher or lower weight on capital α?

Steady-state

- Is there some $\overline{k}$ that solves $\overline{k} = g(\overline{k})$?

$$\overline{k} = \frac{1-\delta}{(1+\gamma_A)(1+\gamma_N)}\overline{k} + \frac{\xi}{(1+\gamma_A)(1+\gamma_N)} f(\overline{k})$$

- Rearranging and using $\gamma_A \gamma_N \approx 0$

$$\overline{k} = \frac{\xi}{\gamma_A + \gamma_N + \delta} f\left(\overline{k}\right) \qquad \text{[Fixed point equation]}$$

- In the case that $f(\overline{k}) = \overline{k}^{\alpha}$: [Function only of parameters! $\{\xi, \gamma_A, \gamma_N, \delta, \alpha\}$]

$$\overline{k} = \frac{\xi}{\gamma_A + \gamma_N + \delta}\overline{k}^{\alpha} \qquad \rightarrow \qquad \overline{k} = \left(\frac{\xi}{\gamma_A + \gamma_N + \delta}\right)^{\frac{1}{1-\alpha}}$$

- *Comparative statics*
  - Increasing in ξ, decreasing in γ_A, γ_N, δ.
  - Convex in ξ, less so if α is smaller $\rightarrow$ Increasing in α

Transition dynamics around steady state

- Is it stable?

$$k_{t+1} = g(k_t) = \frac{1-\delta}{(1+\gamma_A)(1+\gamma_N)} k_t + \frac{\xi}{(1+\gamma_A)(1+\gamma_N)} f(k_t)$$

- Linearize around steady state

$$k_{t+1} \approx g(\bar{k}) + g'(\bar{k})(k_t - \bar{k})$$

- Using $\bar{k} = g(\bar{k})$

$$\widehat{k}_{t+1} \approx \left\langle \frac{g'(\bar{k})\bar{k}}{g(\bar{k})} \right\rangle \widehat{k}_t \quad , \quad \widehat{k}_t := \frac{k_t - \bar{k}}{\bar{k}}$$

So if $\varepsilon_{g,k} < 1$, then the *gap from steady-state* shrinks $|\hat{k}_{t+1}| < |\hat{k}_t|$.

∗ Show that this is stable *iff* the elasticity $\varepsilon_{f,k} < 1$. What parameter determines this? Conclude that growth $\widehat{k}_t$ is higher the further away from steady-state. Compare U.S. vs. Japan after WWII

Transition dynamics around steady state

- How is growth related to distance from steady state?

$$k_{t+1} = g(k_t) = \frac{1-\delta}{(1+\gamma_A)(1+\gamma_N)} k_t + \frac{\xi}{(1+\gamma_A)(1+\gamma_N)} f(k_t)$$

- Linearize around k_t

$$k_{t+1} \approx g(k_t) + g'(k_t)(k_{t+1} - k_t)$$

- Using $k_{t+1} = g(k_t)$ and $k_t = g(k_{t-1})$

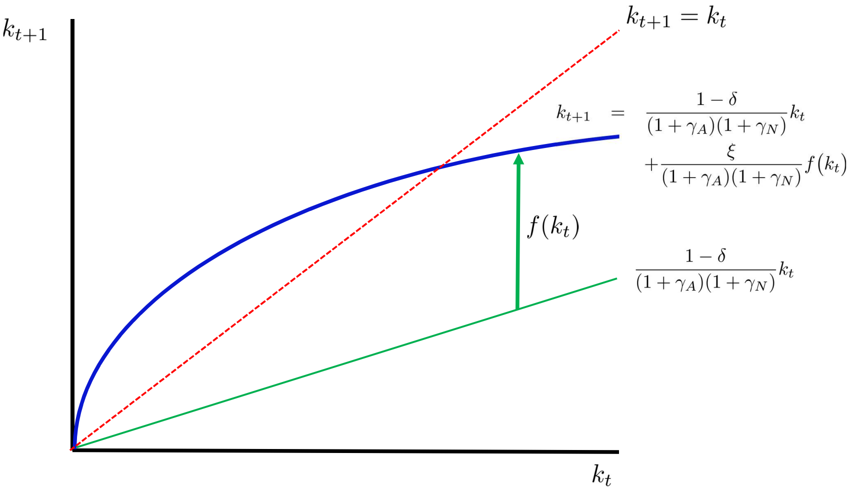
$$\Delta \log k_{t+1} \approx \left\langle \frac{g'(k_t) k_t}{g(k_t)} \right\rangle \Delta \log k_t \quad , \quad \Delta \log k_{t+1} \approx \frac{k_{t+1} - k_t}{k_t}$$

So if $\varepsilon_{g,k} < 1$, then *growth rate* declines: $|\Delta \log k_{t+1}| < |\Delta \log k_t|$.

∗ Show that if $y_t = k_t^\alpha$, then $|\Delta \log y_{t+1}| < |\Delta \log y_t|$.

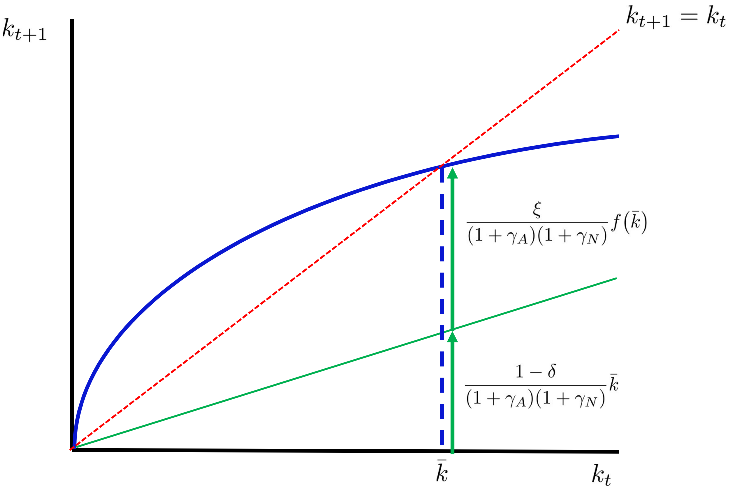
Capital per worker

Properties of function $k_{t+1} = g(k_t)$.



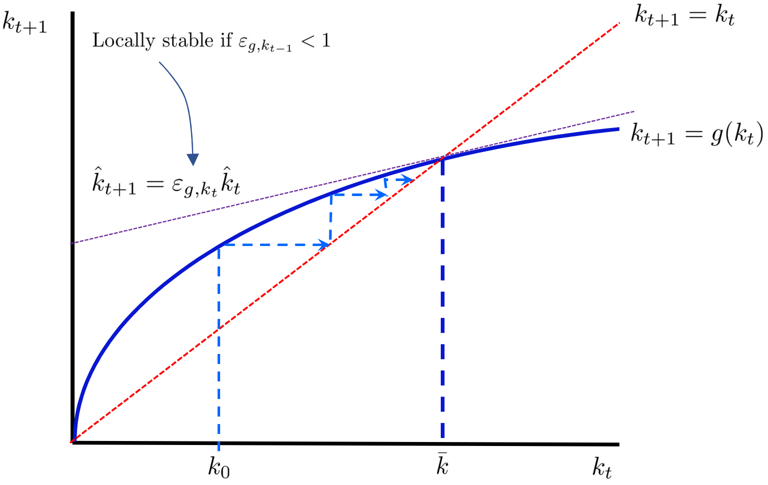
Steady state

Properties of function $k_{t+1} = g(k_t)$. Steady state $\bar{k}$ such that $\bar{k} = g(\bar{k})$.



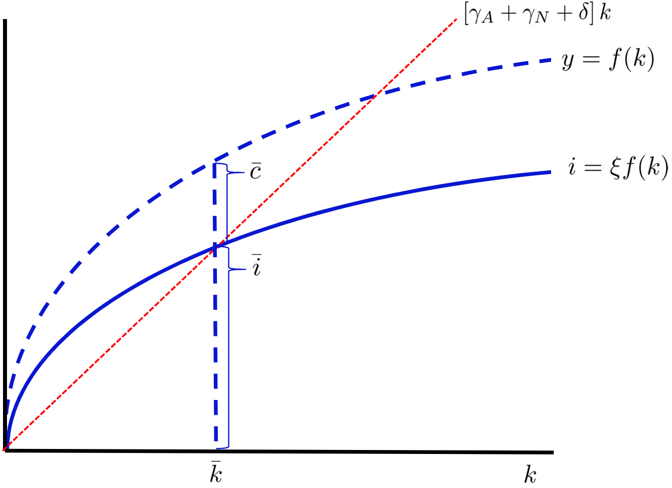
Transition dynamics

Properties of function $k_{t+1} = g(k_t)$. From a lower k_0, the growth rate $\hat{k}_t$ is higher. System is stable: k_t converges to $\bar{k}$.



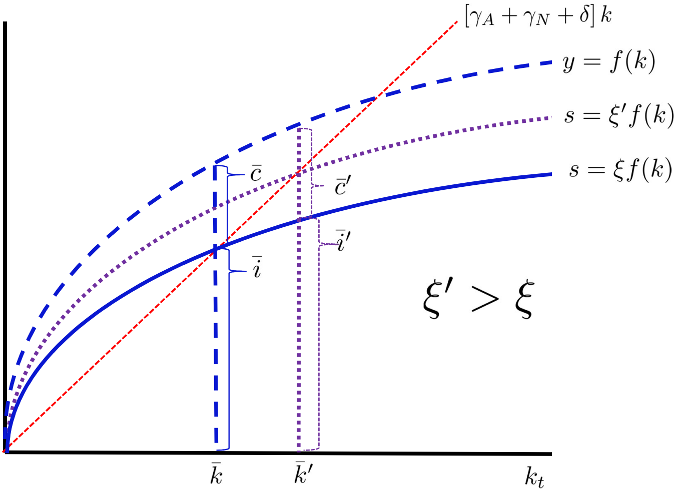
Steady state - Intuition

Steady state $\bar{k}$ is such that total savings per worker when $y = f(\bar{k})$ increases capital per worker by enough to account for population growth, productivity growth and deprectiation.



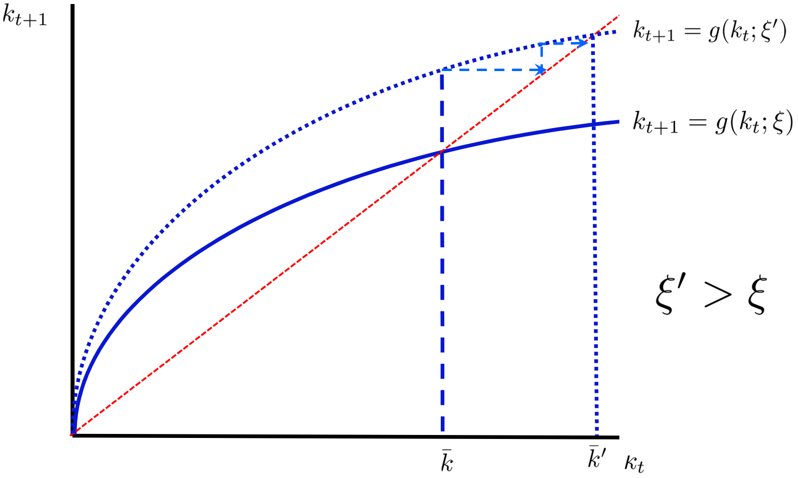
Dynamics following a shock

Starting in steady-state there is a one-time permanent increase in ξ to $\xi' > \xi$. New steady-state value $\bar{k}'$ increases.



Dynamics following a shock

Starting in steady-state there is a one-time permanent increase in ξ to $\xi' > \xi$. New steady-state value $\bar{k}'$ increases. k_t grows quickly then slows as it nears new steady-state.



Positive implications - Growth facts

1. Share of income to labor is constant and W_t grows at constant rate

$$\frac{W_t N_t}{Y_t} = 1-\alpha \quad , \quad \frac{W_{t+1}}{W_t} = \frac{(1-\alpha)Y_{t+1}/N_{t+1}}{(1-\alpha)Y_t/N_t} = \gamma_A$$

2. Capital-Output ratio is constant

$$\frac{K_t}{Y_t} = \frac{K_t/A_t N_t}{Y_t/A_t N_t} = \frac{\overline{k}}{f(\overline{k})}$$

3. Capital per worker and output per worker grow at constant rates

$$\frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{A_{t+1}}{A_t} \times \frac{K_{t+1}/A_{t+1}N_{t+1}}{K_t/A_t N_t} = \gamma_A \frac{\overline{k}}{\overline{k}} = \gamma_A$$

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \cdots = \left(\frac{A_{t+1}}{A_t}\right)^{\alpha} \times \left(\frac{K_{t+1}/N_{t+1}}{K_t/N_t}\right)^{1-\alpha} = \gamma_A$$

4. Capital grows at a constant rate

$$\frac{K_{t+1}}{K_t} = \cdots = (1+\gamma_A)(1+\gamma_N)$$

Positive implications - Growth facts

1. Share of income to labor is constant and W_t grows at constant rate

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4. Capital grows at a constant rate

$$\frac{K_{t+1}}{K_t} = \cdots = (1+\gamma_A)(1+\gamma_N)$$

$*$ $\gamma_A > 0$ key for economic growth! Not ξ or γ_N

Normative implications - Optimal savings

- What value $\overline{k}^*$ maximizes steady-state consumption per worker

$$\overline{c}^* = \max_{\overline{k}} \; \overline{y}\left(\overline{k}\right) - \overline{i}\left(\overline{k}\right) = \max_{\overline{k}} \; f\left(\overline{k}\right) - \underbrace{\left[(1+\gamma_A)(1+\gamma_N) - (1-\delta)\right]}_{\approx \gamma_A + \gamma_N + \delta} \overline{k}$$

- The *golden rule level of capital* satisfies

$$\overline{k}: \qquad f'\left(\overline{k}^*\right) = (\gamma_A + \gamma_N + \delta)$$

- In the Cobb-Douglas case: $\varepsilon_{f,k} = \alpha$

$$\frac{f'\left(\overline{k}^*\right)\overline{k}^*}{f\left(\overline{k}^*\right)} = (\gamma_A + \gamma_N + \delta)\frac{\overline{k}^*}{f\left(\overline{k}^*\right)} \implies \overline{k}^* = \left(\frac{\alpha}{\gamma_A + \gamma_N + \delta}\right) f(\overline{k}^*)$$

- Recall that the steady state level of capital is given by

$$\overline{k} = \left(\frac{\xi}{\gamma_A + \gamma_N + \delta}\right) f\left(\overline{k}\right)$$

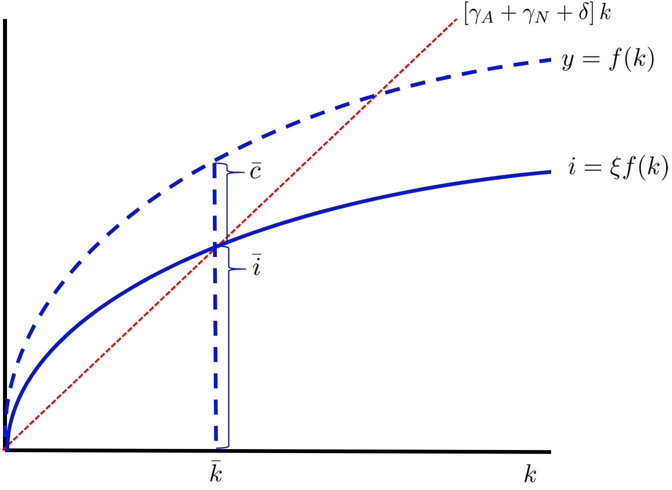
- $\therefore$ the golden rule level of savings ξ^* that implements $\overline{k}^*$ is

$$\xi^* = \alpha$$

- *Share of income saved and invested = Share of income paid to capital*

Golden rule

With a saving rate of ξ^*, the steady-state gives the golden rule level of capital $\bar{k}^*$. This satisfies the condition $f'(\bar{k}^*) = (\gamma_A + \gamma_N + \delta)$. Note that $\bar{c}^* > \bar{c}$!



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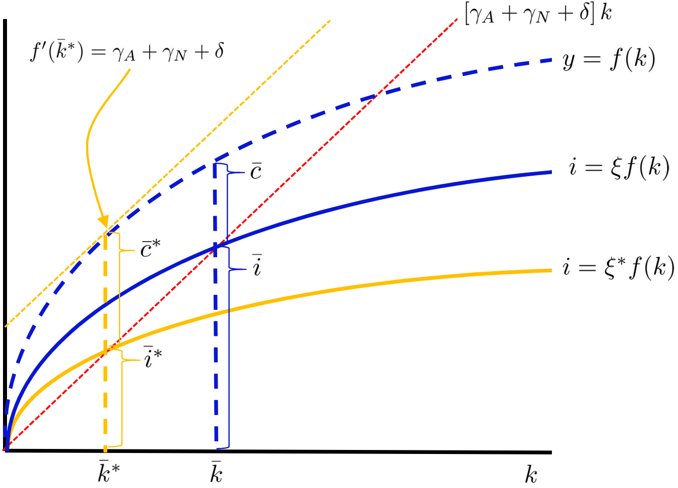
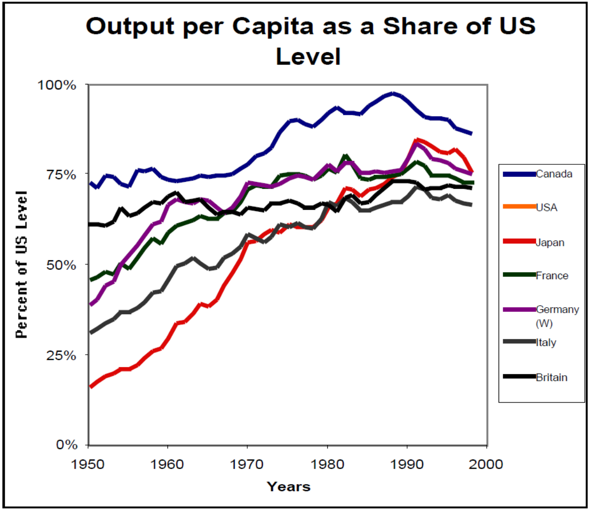


Data

Are countries all converging to the same $\overline{k}$?



Data

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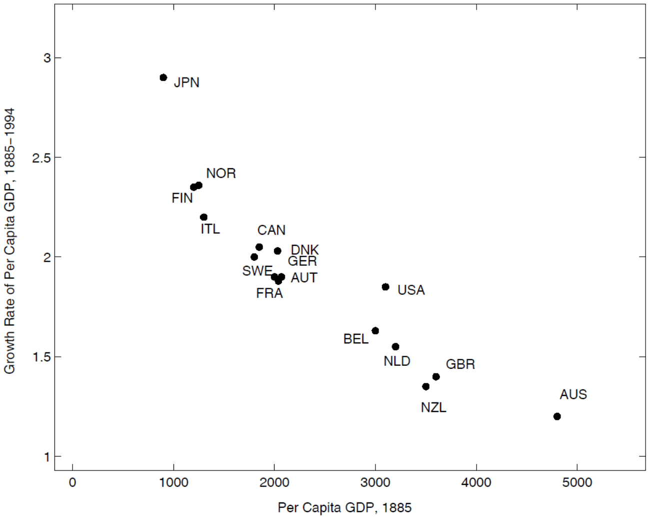
Figure 1.a: Growth Rate Versus Initial Per Capita GDP



Data

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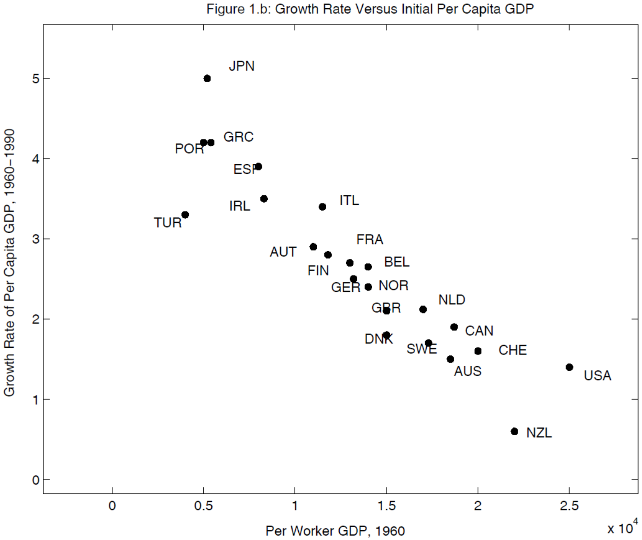
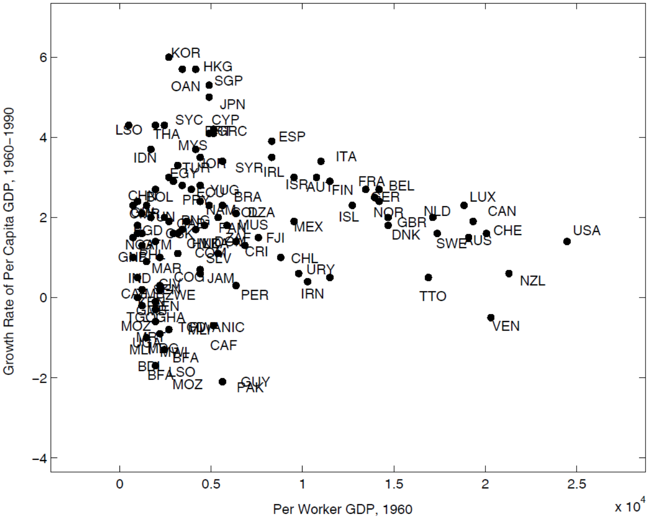


Data

Are countries all converging to the same $\bar{k}$?

Figure 1.c: Growth Rate Versus Initial Per Capita GDP



Covered

- Growth facts

- Steady state comparative statics, Transition dynamics

  - How does the steady state level of $\overline{k}$ depend on the parameters of the economy $\{\xi, \gamma_A, \gamma_N, \alpha, \delta\}$?

  - From an initial $k_0 \lessgtr \overline{k}$ what are the dynamics of k_t?

  - From an initial $k_0 = \overline{k}$, if a parameter changes, what happens to k_t?

  - How do these answers depend on parameters?

- *Next*

  - TA - More practice linearization
  - Lecture - Endogenizing savings ... the neo-classical growth model.