Honors - Economic Analysis III

Lecture 3: Neoclassical model

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This lecture

Issues with Solow model

Neoclassical model

- Ramsey (1928), Cass (1965), Koopmans (1965)

Optimality conditions

Steady state + Transition dynamics

Next: Decentralized economy, Welfare Theorems, Government

Recap of Solow model

- Equilibrium conditions (without growth!)

$$\begin{aligned} y_t &= Af(k_t) = Ak_t^{\alpha} \\ y_t &= c_t + i_t \\ k_{t+1} &= (1-\delta)k_t + \xi f(k_t) \\ c_t &= (1-\xi)y_t \end{aligned}$$

- Solution

$$k_{t+1} = g(k_t)$$

- Steady state

$$\overline{k} = g(\overline{k}) \quad , \quad \overline{c} = f(\overline{k}) - \delta \overline{k} \quad \longrightarrow \text{Golden rule} \longrightarrow \quad \left(\overline{k}^*, \overline{c}^*\right)$$

- Transition dynamics

$$\widehat{k}_{t+1} = \frac{\partial g(\overline{k})}{\partial \overline{k}} \frac{k}{} \widehat{k}_t \quad , \quad \widehat{c}_t = \frac{\partial f(\overline{k})}{\partial \overline{k}} \frac{k}{} \widehat{k}_t = \alpha \hat{k}_t$$

Issues with Solow model

- Positive (descriptive)
  - Within country over-time differences in $\widehat{y}_t$
  - Cross country differences in $\widehat{y}_t$ as functions of parameters $\{\xi, \delta, \alpha, A\}$
- Normative
  - Households don't optimize $\rightarrow$ Bad for policy
  - E.g. If A increases for a few periods, shouldn't ξ respond? Maybe households would like to save some of a temporary increase in their income?

Issues with Solow model

- Neoclassical model
  - Allow households to make savings decisions
  - Control the division of output into consumption and investment

$$
\begin{aligned}
c_t + i_t &= f(k_t) \\
k_{t+1} &= (1-\delta)k_t + i_t
\end{aligned}
$$

  - Economy of optimizing individuals $\rightarrow$ Good for policy
  - Time series of endogenous variables $\{c_t, y_t, k_t, i_t\}_{t=0}^{\infty}$ determined by parameters and initial conditions.
  - Keep other features of Solow $\rightarrow$ Still match growth facts!

- Work horse model in modern macroeconomics
  - ✓ + Welfare - Centralized vs. Decentralized economy
  - ✓ + Labor supply
  - ✓ + Stochastic processes for A_t. Real Business Cycle model
  - ✓ + Heterogeneity. Study inequality.
  - × + Fiscal policy - 'Ramsey problem'
  - × + Monetary policy - 'New Keynesian models'. 'Nominal' Business Cycle model

Environment - Centralized

- *Time* - Discrete $t = 0, 1, 2 \ldots$
- *Agents* - Representative household with $\overline{N}$ identical workers
- *Goods* - One good can either be used for consumption or investment

$$C_t + I_t = Y_t$$

- *Preferences* - Utility of the household at date 0 is

$$\sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \quad , \quad \beta \in (0,1) \quad , \quad U(C_t, N_t) = u(C_t/N_t)$$

- *Technology* - Constant returns to scale production technology

$$Y_t = AF(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha}.$$

Capital depreciates at rate δ

$$K_{t+1} = (1-\delta)K_t + I_t \quad , \quad \delta \in [0,1] \quad , \quad K_0 > 0$$

Environment - Centralized

- *Time* - Discrete $t = 0, 1, 2\dots$
- *Agents* - Representative household
- *Goods* - One good can either be used for consumption or investment

$$c_t + i_t = y_t$$

- *Preferences* - Utility of the household at date 0 is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \underbrace{\beta \in (0,1)}_{\text{Rate of time preference}}$$

- *Technology* - Constant returns to scale production technology

$$y_t = Af(k_t) = Ak_t^{\alpha}.$$

Capital depreciates at rate δ

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Environment - Centralized

- *Time* - Discrete $t = 0, 1, 2 \ldots$
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- *Goods* - One good can either be used for consumption or investment

$$c_t + i_t = y_t$$

- *Preferences* - Utility of the household at date 0 is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \ = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) \cdots + \beta^t u(c_t) + \ldots$$

- *Technology* - Constant returns to scale production technology

$$y_t = Af(k_t) = Ak_t^{\alpha}.$$

Capital depreciates at rate δ

$$k_{t+1} = (1-\delta)k_t + i_t \quad , \quad \delta \in [0,1] \quad , \quad k_0 > 0$$

Problem

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the series of constraints

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t \quad , \quad t = 0, 1, 2, \ldots$$

and initial conditions

$$k_0 > 0$$

Problem - Lagrangean

- Constrained optimization problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) + (1-\delta) k_t - c_t - k_{t+1} \right]$$

- First order necessary conditions

$$\begin{aligned} c_t: \quad 0 &= \beta^t u'(c_t) - \lambda_t \\ k_{t+1}: \quad 0 &= -\lambda_t + \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] \end{aligned}$$

- Combining conditions

$$\begin{aligned} 0 &= -\beta^t u'(c_t) + \beta^{t+1} u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right] \\ u'(c_t) &= \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right] \end{aligned}$$

Euler equation

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Interpretation - *Perturbational method*
    - Suppose we have the optimal $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$
    - Reduce consumption by ε units only today, i.e. at date t
    - Loss in utility today is approx. $u'(c_t^*)\varepsilon$ $\quad u(c_t) \approx u(c_t^*) + u'(c_t^*)(c_t - c_t^*)$
    - Plan: Take ε, invest it in capital, consume proceeds tomorrow
    - Two effects tomorrow (i) increase output by $f'(k_{t+1}^*)\varepsilon$, (ii) increase capital by $(1-\delta)\varepsilon$
    - Gain in utility tomorrow approx. $u'(c_{t+1}^*)[f'(k_{t+1}^*) + (1-\delta)]\varepsilon$
    - If $\{c_t^*, k_{t+1}^*\}$ is optimal, then should be no change in total utility:

$$0 = -u'(c_t^*)\varepsilon + \beta u'(c_{t+1}^*) \left[f'(k_{t+1}^*) + (1-\delta) \right] \varepsilon$$

Partial solution

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1 - \delta) \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1 - \delta) k_t$$

- Initial conditions

$$k_0 > 0$$

Transversality condition

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{T}$ Truncated at $T < \infty$

$$\max \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to the series of constraints

$$\begin{aligned}
c_0 + k_1 &\leq f(k_0) + (1-\delta)k_0 \quad , \quad k_0 > 0 \\
c_1 + k_2 &\leq f(k_1) + (1-\delta)k_1 \\
&\dots \\
c_T + k_{T+1} &\leq f(k_T) + (1-\delta)k_T
\end{aligned}$$

Issue - Household wants k_{T+1} infinitely negative

Problem

Household chooses sequences of $\{c_t, k_{t+1}\}_{t=0}^{T}$

$$\max \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to the series of constraints

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad t = 0, 1, 2, \ldots, T$$

initial conditions

$$k_0 > 0$$

and a non-negativity constraint on capital

$$k_{t+1} \geq 0$$

Transversality condition

- Constrained optimization problem

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t \Big[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \Big] + \sum_{t=0}^{T} \mu_t k_{t+1}$$

- First order necessary conditions

$$\begin{aligned} c_t : \quad 0 &= \beta^t u'(c_t) - \lambda_t \\ k_{t+1} : \quad 0 &= -\lambda_t + \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] + \mu_t \\ k_{T+1} : \quad 0 &= -\lambda_T + \mu_T \end{aligned}$$

- Multipliers and constraints

$$\begin{aligned} \lambda_t \geq 0 \quad , \quad 0 &= f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \\ \mu_t \geq 0 \quad , \quad 0 &\leq k_{t+1} \end{aligned}$$

- Complementary slackness conditions

$$\begin{aligned} 0 &= \lambda_t \Big[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \Big] \\ 0 &= \mu_t \Big[k_{t+1} \Big] \end{aligned}$$

Transversality condition

- Optimal c_T

$$\lambda_T = \beta^T u'(c_T)$$

- Optimal k_{T+1}

$$\mu_T = \lambda_T = \beta^T u'(c_T)$$

- Complementary slackness

$$\beta^T u'(c_T) k_{T+1} = 0$$

- *The present discounted utility value of 'left over' resources must be equal to zero*
- We can generalize this to the infinite horizon case

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

Example - A 'Cake-eating' problem

- Constrained optimization problem

$$\max_{\{c_t\}_{t=0}^{\infty}} = \sum_{t=0}^{\infty} \beta^t c_t$$

subject to

$$W_{t+1} = R(W_t - c_t) \quad \forall t \quad , \quad W_0 > 0 \quad , \quad W_{t+1} \geq 0$$

- Suppose $\beta R < 1$, then eat the whole cake today. This is fine.
- Suppose $\beta R > 1$, then defer consumption every period. This seems weird.

  ... as $t \to \infty$ we never eat the cake?
- Transversality condition - PDV of future cake has to be zero

$$\lim_{T \to \infty} \beta^T W_{T+1} = 0$$

Full solution

- Initial conditions

$$k_0 > 0$$

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Transversality condition

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

Characterizing the solution

1. Steady state

- Solve $(\bar{c}, \bar{k})$ as a function of parameters of the model
- If we change a parameter in the economy, how does the steady state change?

  $\rightarrow$ *Steady state comparative statics*

2. Dynamics

- Is the steady state *globally stable* ?
- What determines the *local dynamics* around steady state?
- From steady state, if we change a parameter, how does the economy evolve?

  $\rightarrow$ *Dynamic comparative statics*

- PS2 - Solution for $u(c) = \log(c)$, $\delta = 1$, $f(k) = Ak^{\alpha}$

1. Steady state

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

1. Steady state

- Euler equation

$$u'(\overline{c}) = \beta u'(\overline{c}) \left[f'(\overline{k}) + (1-\delta) \right]$$

- Resource constraint

$$\overline{c} + \overline{k} = f(\overline{k}) + (1-\delta)\overline{k}$$

1. Steady state

- Euler equation

$$u'(\overline{c}) = \beta u'(\overline{c}) \left[f'(\overline{k}) + (1-\delta) \right]$$

- Resource constraint

$$\overline{c} + \overline{k} = f(\overline{k}) + (1-\delta)\overline{k}$$

Golden-rule? Now redundant

- There are no left over ‘policy variables’ like the savings rate for the planner to choose.

1. Steady state

- Euler equation

$$u'(\overline{c}) = \beta u'(\overline{c}) \left[f'(\overline{k}) + (1-\delta) \right]$$

- Resource constraint

$$\overline{c} + \overline{k} = f(\overline{k}) + (1-\delta)\overline{k}$$

Golden-rule? Now redundant

- There are no left over ‘policy variables’ like the savings rate for the planner to choose.

- Monetary economics (Prof. Uhlig) - Policy parameters return!
    - Central bank rule for nominal interest rates: $\hat{i}_t = \phi_y \widehat{y}_t + \phi_\pi \pi_t$

1. Steady state

- Euler equation

$$f'(\overline{k}) = \frac{1-\beta}{\beta} + \delta \qquad \text{Solow Golden Rule:} \quad f'(\overline{k}^*) = \delta \implies \overline{k} < \overline{k}^*$$

- Resource constraint

$$\overline{c} = f(\overline{k}) - \delta \overline{k}$$

Comparative statics

$\downarrow \beta$ Value future output relatively less so $\downarrow \overline{k}$.
Also lower $\downarrow \overline{c}$.

$\uparrow \delta$ Wasted savings, requires higher $\uparrow f'(\overline{k})$, so $\downarrow \overline{k}$.
Also lower $\downarrow \overline{c}$

1. Steady state

- Euler equation

$$f'(\overline{k}) = \frac{1-\beta}{\beta} + \delta \qquad \text{Solow Golden Rule:} \quad f'(\overline{k}^*) = \delta \implies \overline{k} < \overline{k}^*$$

- Resource constraint

$$\overline{c} = f(\overline{k}) - \delta \overline{k}$$

Comparative statics

$\downarrow \beta$ Value future output relatively less so $\downarrow \overline{k}$.
Also lower $\downarrow \overline{c}$.

$\uparrow \delta$ Wasted savings, requires higher $\uparrow f'(\overline{k})$, so $\downarrow \overline{k}$.
Also lower $\downarrow \overline{c}$
Why is $\overline{c}$ increasing in $\overline{k}$ around $(\overline{c}, \overline{k})$?

2. Dynamics - Phase diagram

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- No change in consumption: $c_{t+1} = c_t$

$$1 = \beta \left[f'(k_t) + (1-\delta) \right] \quad \rightarrow \quad k_t = \overline{k}$$

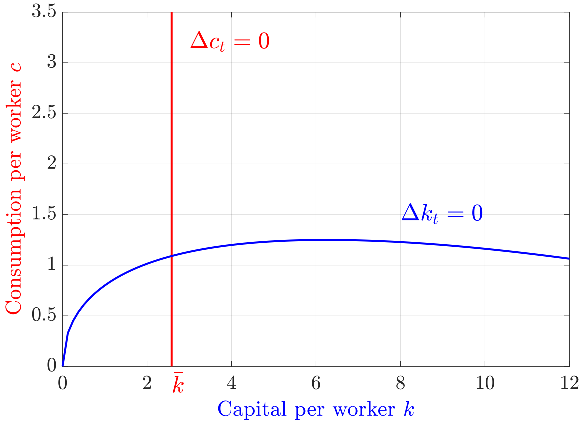
- No change in capital: $k_{t+1} = k_t$

$$c_t = f(k_t) - \delta k_t$$

2. Dynamics - Phase diagram

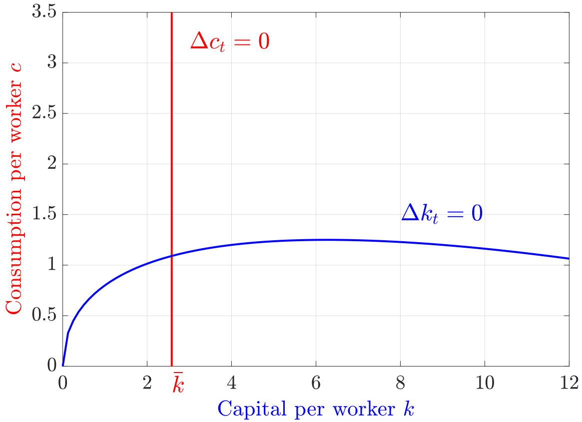
$$\Delta c_t = 0: \qquad k_t = \bar{k}$$

$$\Delta k_t = 0: \qquad c_t = f(k_t) - \delta k_t$$



2. Dynamics - Phase diagram

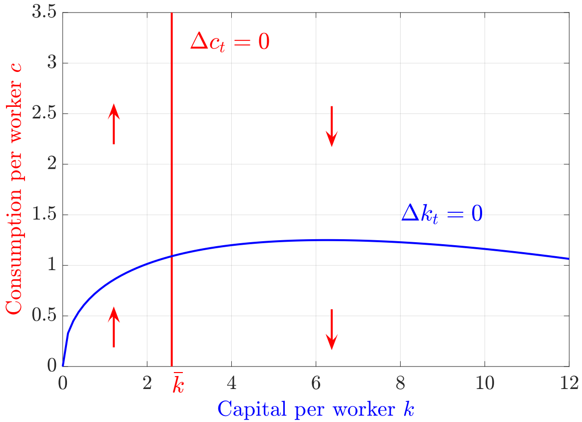
$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right] \\
c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t
\end{aligned}
$$



2. Dynamics - Phase diagram

$$u'(c_t) \quad = \quad \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]$$

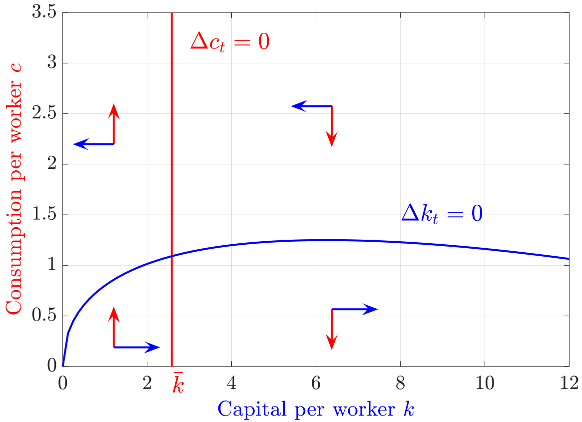
If $k_{t+1} > \bar{k}$, then low MPK_{t+1}, so $\uparrow c_t, \downarrow c_{t+1}$ so consumption is <u>falling</u>.



2. Dynamics - Phase diagram

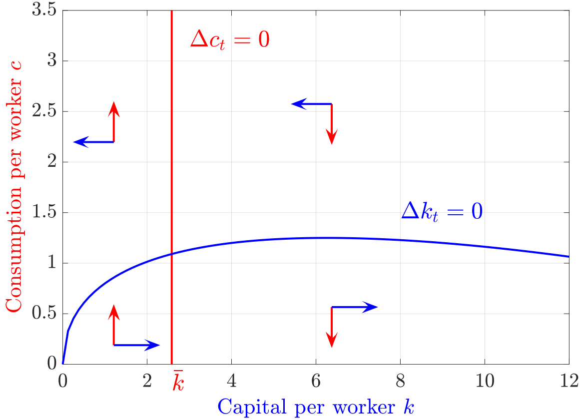
$$c_t + k_{t+1} \quad = \quad f(k_t) + (1-\delta)k_t$$

If $c_t > \bar{c}(k_t)$, then consuming more than $f(k_t) - \delta k_t$, so capital is falling.



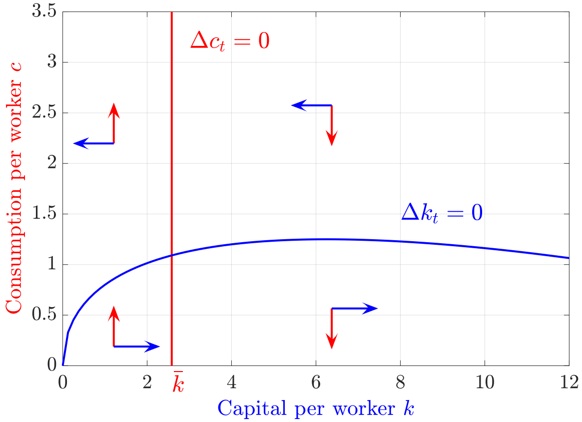
2. Dynamics - Phase diagram

- To the N.W. (↖) we violate the *resource constraint*
- Increasing marginal product of capital, increasing consumption, at some point $c_t > f(k_t)$



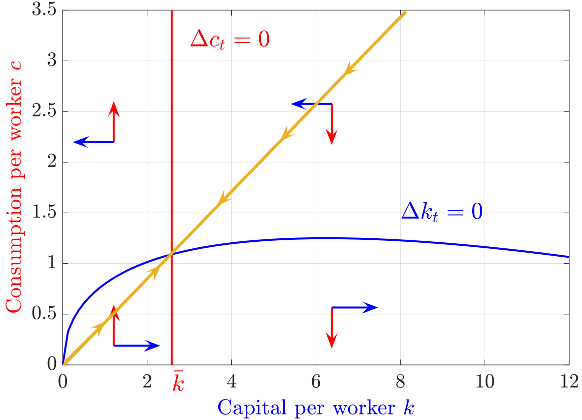
2. Dynamics - Phase diagram

- To the S.E. ($\searrow$) we violate the *transversality consition*
- Decreasing marginal product of capital, falling consumption, $u'(c_t) \to \infty$ despite accumulating capital



2. Dynamics - Saddle path

- On the *saddle-path* all equilibrium conditions hold
- Economy converges to steadv-state



2. Dynamics - Comparative statics

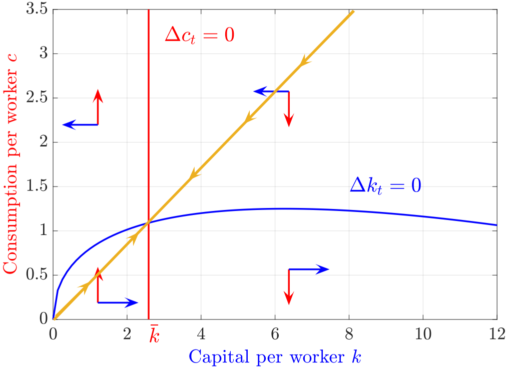
- If initially in steady-state and we *permanently* change a parameter

(i) What changes the $\Delta c_t = 0$ and $\Delta k_t = 0$ lines?

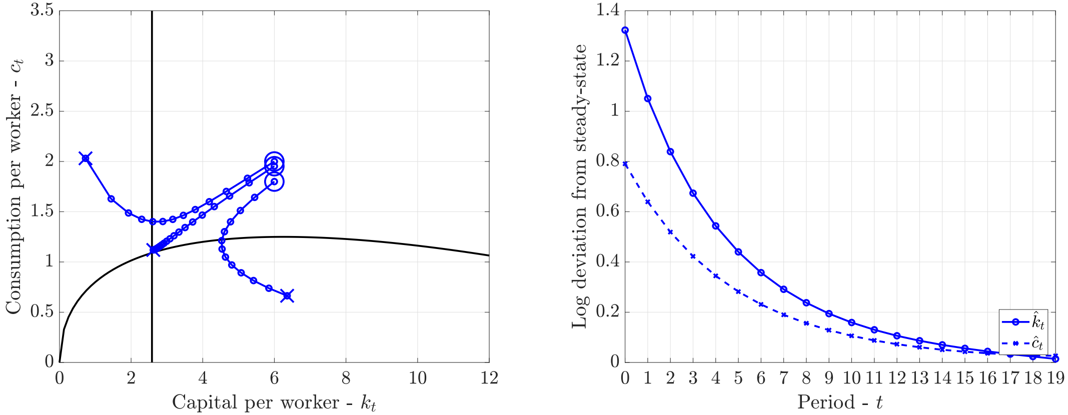
(ii) Is there a new saddle path?

(iii) Consumption jumps to new saddle path and economy converges (at a decreasing rate) toward steady-state.

- Should be able to describe this behaviour in terms of the location of the new steady state (why has $\bar{k}$ increased? decreased?), *and* optimal capital accumulation decision of the household along the transition path.

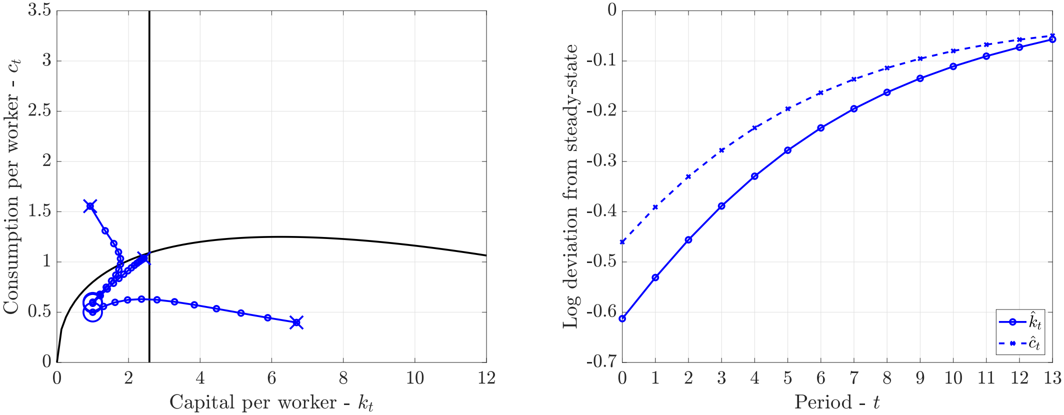


2. Dynamics - Saddle path



Parameters - $\alpha = 0.50,\ \delta = 0.20,\ \beta = 0.90,\ f(k) = Ak^{0.50},\ u(c) = \log c$

2. Dynamics - Saddle path



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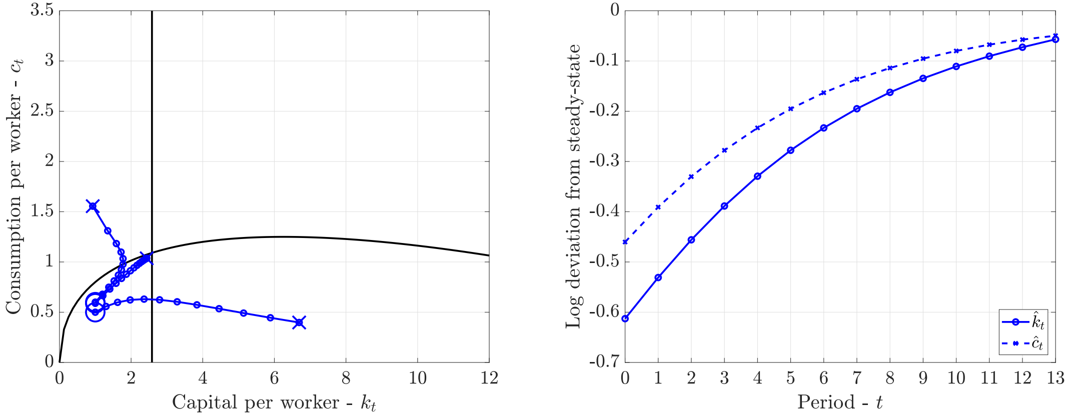
2. Dynamics - Saddle path



Parameters - $\alpha = 0.50$, $\delta = 0.20$, $\beta = 0.90$, $f(k) = Ak^{0.50}$, $u(c) = \log c$

Can we say something about what governs how quickly the economy converges?

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta)k_{t+1} \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation - $\delta = 1$

$$\left\{ \frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} \right\} \widehat{c}_t = \left\{ \frac{\beta u''(\bar{c}) f'(\bar{k}) \bar{c}}{u'(\bar{c})} \right\} \widehat{c}_{t+1} + \left\{ \frac{\beta u'(\bar{c}) f''(\bar{k}) \bar{k}}{u'(\bar{c})} \right\} \widehat{k}_{t+1}$$

- Resource constraint

$$\left\{ \frac{1\bar{k}}{\bar{k}} \right\} \widehat{k}_{t+1} = \left\{ \frac{f'(\bar{k})\bar{k}}{\bar{k}} \right\} \widehat{k}_t - \left\{ \frac{\bar{c}}{\bar{k}} \right\} \widehat{c}_t$$

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) k_{t+1} \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta) k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation

$$\left\{ \frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} \right\} \widehat{c}_t = \left\{ \frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} \right\} \widehat{c}_{t+1} + \left\{ \frac{f''(\bar{k})\bar{k}}{f'(\bar{k})} \right\} \beta f'(\bar{k}) \widehat{k}_{t+1}$$

- Resource constraint

$$\left\{ \frac{\bar{k}}{\bar{\bar{k}}} \right\} \widehat{k}_{t+1} = \left\{ \frac{\bar{k}}{\beta \bar{\bar{k}}} \right\} \widehat{k}_t - \left\{ \frac{\bar{c}}{\bar{\bar{k}}} \right\} \widehat{c}_t$$

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) k_{t+1} \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta) k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation

$$\left\{ \frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} \right\} \hat{c}_t = \left\{ \frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} \right\} \hat{c}_{t+1} + \left\{ \frac{f''(\bar{k})\bar{k}}{f'(\bar{k})} \right\} \hat{k}_{t+1}$$

- Resource constraint

$$\left\{ \frac{\bar{k}}{\bar{\bar{k}}} \right\} \hat{k}_{t+1} = \left\{ \frac{\bar{k}}{\beta \bar{\bar{k}}} \right\} \hat{k}_t - \left\{ \frac{\bar{c}}{\bar{\bar{k}}} \right\} \hat{c}_t$$

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation - $u'(c) \propto c^{-\sigma}$, $f'(k) \propto k^{\alpha-1}$

$$-\sigma \widehat{c}_t = -\sigma \widehat{c}_{t+1} + (\alpha - 1)\widehat{k}_{t+1}$$

- Resource constraint

$$\widehat{k}_{t+1} = \frac{1}{\beta}\widehat{k}_t - \frac{\overline{c}}{\overline{k}}\widehat{c}_t$$

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation - $u'(c) \propto c^{-\sigma}$, $f'(k) \propto k^{\alpha-1}$

$$\widehat{c}_{t+1} - \widehat{c}_t = \left\{ -\frac{1-\alpha}{\sigma} \right\} \widehat{k}_{t+1}$$

1. $\uparrow \sigma$ more curvature in utility function

- Smoother consumption $\rightarrow$ Smaller rate of change in $\widehat{c}_t$ for any $\widehat{k}_t$

2. Dynamics - Local dynamics

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

- Resource constraint

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

- *Local* approximation of equilibrium conditions around steady-state
- Euler equation - $u'(c) \propto c^{-\sigma}$, $f'(k) \propto k^{\alpha-1}$

$$\widehat{c}_{t+1} - \widehat{c}_t = \left\{ -\frac{1-\alpha}{\sigma} \right\} \widehat{k}_{t+1}$$

2. $\downarrow \alpha$ more curvature in production

- Marginal product of capital more responsive to $\widehat{k}_t$
- Consumption changes more quickly in response to $\widehat{k}_t \neq 0$

2. Dynamics - Local dynamics

- Rearrange these (use R.C. to sub $\widehat{k}_{t+1}$ in E.E.)

$$\begin{bmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{bmatrix} = A \begin{bmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \end{bmatrix} \quad \text{which implies that} \quad \begin{bmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \end{bmatrix} = A^t \begin{bmatrix} \widehat{c}_{0} \\ \widehat{k}_{0} \end{bmatrix}$$

- Eigen-vector decomposition $Av_1 = \lambda_1 v_1 \to AV = V\Lambda$

$$\begin{bmatrix} \widehat{c}_{2} \\ \widehat{k}_{2} \end{bmatrix} = \underbrace{V\Lambda V^{-1}}_{A}\underbrace{V\Lambda V^{-1}}_{A} \begin{bmatrix} \widehat{k}_{0} \\ \widehat{c}_{0} \end{bmatrix} = V\Lambda^2 V^{-1} \begin{bmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \end{bmatrix}$$

- In the limit as $t \to \infty$

$$\lim_{t\to\infty} \begin{bmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \end{bmatrix} = \lim_{t\to\infty} V \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} V^{-1} \begin{bmatrix} \widehat{c}_{0} \\ \widehat{k}_{0} \end{bmatrix}$$

- Macroeconomics models will deliver $0 < \lambda_1 < 1 < \lambda_2$
  - Why? Discounting $\beta < 1$, Concavity $u'' < 0$, $f'' < 0$
  - See: PS2 Q5(c)

2. Dynamics - Local dynamics

- Macro problems will deliver $0 < \lambda_1 < 1 < \lambda_2$
- Now we can write an expression for k_t and c_t, using $A^t = V\Lambda^t V^{-1}$
- Recall the simple rule for the inverse of a 2×2 matrix!

$$\begin{bmatrix} \widehat{c}_t \\ \widehat{k}_t \end{bmatrix} = \frac{1}{det(V)} \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \begin{bmatrix} v_{22} & -v_{21} \\ -v_{12} & v_{11} \end{bmatrix} \begin{bmatrix} \widehat{c}_0 \\ \widehat{k}_0 \end{bmatrix}$$

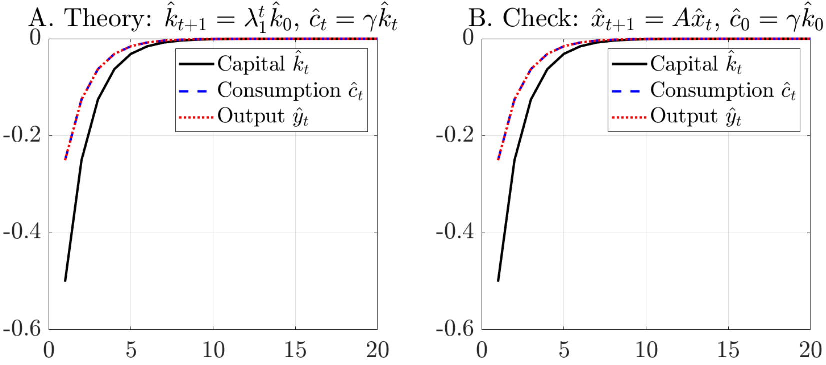
- If $-v_{12}\widehat{c}_0 + v_{11}\widehat{k}_0 \neq 0$, then gets multiplied by the *explosive eigen value* λ_2
- Will continue to propogate, sending $\widehat{k}_t$ and $\widehat{c}_t$ to either infinity or zero!
- Both cases violate equilibrium conditions!
- The *saddle path* consists of a value of $\widehat{c}_0$ such that given $\widehat{k}_0$, these conditions are not violated. This explosive eigen value is killed off.

$$-v_{12}\widehat{k}_0 + v_{11}\widehat{c}_0 = 0 \quad \rightarrow \quad \widehat{c}_0 = (v_{11}/v_{12}) \times \widehat{k}_0$$

- PS2 Q5 - Compute v_{11}, v_{12} by hand for $\delta = 1$, $u(c) = \log c$, $f(k) = k^\alpha$
- Bonus - Show $\widehat{k}_t = \lambda_1^t \widehat{k}_0$. So λ_1 *explicitly* governs rate of convergence(!!!)

2. Dynamics - Local dynamics

- Canvas - Some simple code `NCMEigensolve.m` that produces these figures. Play around with the code!
- What causes faster convergence: $(\downarrow \alpha, \downarrow \sigma)$? Does the RHS plot diverge in line with theory for $\widehat{c}_0 \neq \gamma \widehat{k}_0$? What happens when $\widehat{k}_0 \lessgtr 0$? Write some code to plot λ_1 as a function of β, σ, α, what do you learn? Have fun!

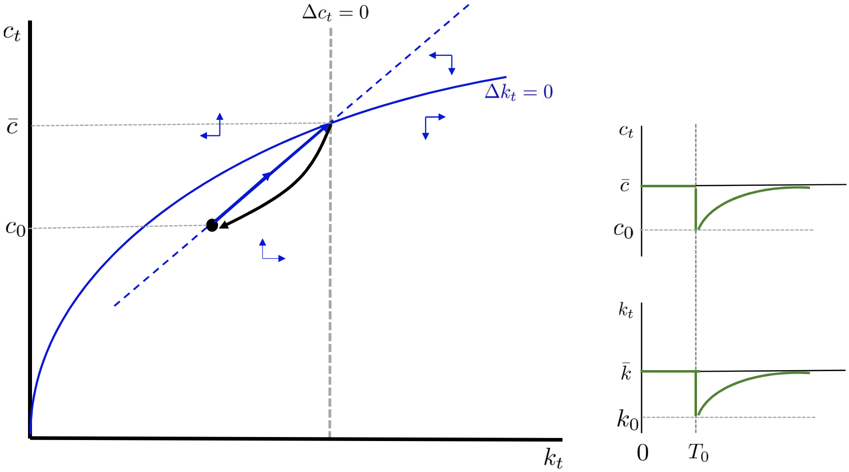


3. Dynamics - Following a shock

- How does the economy respond following a shock?
- What if that shock is permanent or transitory?
- What if it is anticipated or unexpected?
- How might this behavior help us understand what generates business cycles?
  - Recall: Y_t, C_t and I_t all drop during a recession, increase during a boom
- (i) Change in k_0, (ii) Changes in β, (iii) Changes in A

3. Dynamics - $\downarrow k_0$ - Unexpected

$\downarrow c_0$ to reaccumulate capital stock. If $\uparrow \sigma$ or $\uparrow \alpha$, then smaller drop and slower rebuild.



3. Dynamics - Changes in patience β

- Resource constraint

$$c_t + k_{t+1} = Af(k_t) + (1-\delta)k_t + g$$

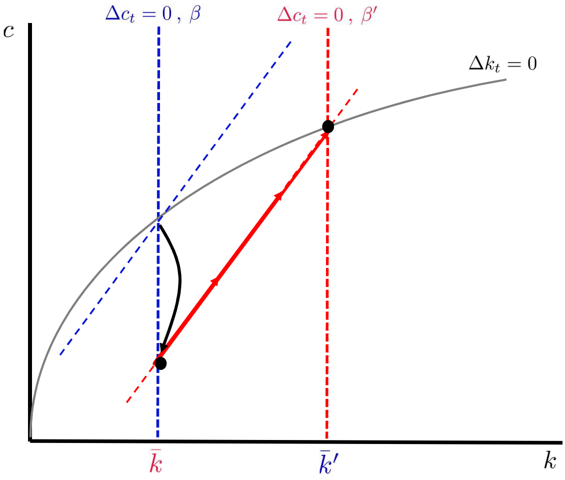
- Euler equation

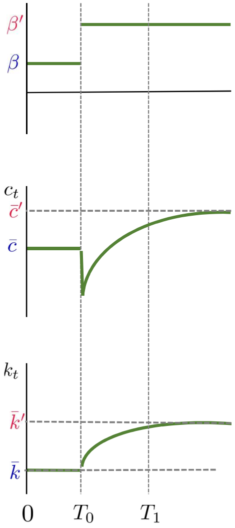
$$u'(c_t) = \beta_t u'(c_{t+1}) \left[Af'(k_{t+1}) + (1-\delta)k_{t+1}\right]$$

- $\Delta c_t = 0$ locus shifts
- $\Delta k_t = 0$ locus is unaffected

3. Dynamics - $\uparrow \beta$ - Unexpected - Permanent

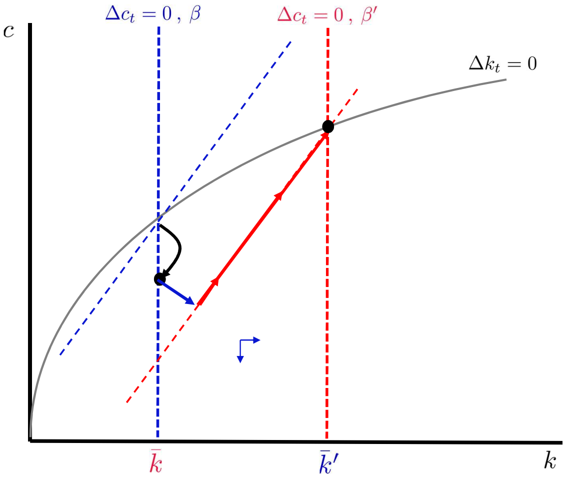
$\downarrow c_0$ to accumulate capital to reach higher steady-state $\overline{k}'$ and $\overline{c}'$.

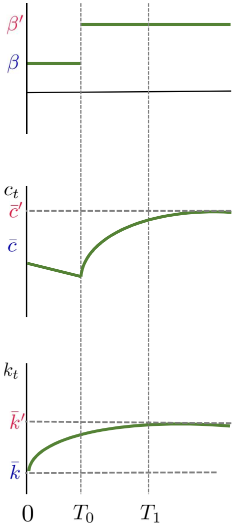




3. Dynamics - $\uparrow \beta$ - Expected - Permanent

$\downarrow c_0$ and accumulate capital to smooth effect on consumption.





3. Dynamics - Changes in productivity A

- Resource constraint

$$c_t + k_{t+1} = A_t f(k_t) + (1-\delta)k_t + g$$

- Euler equation

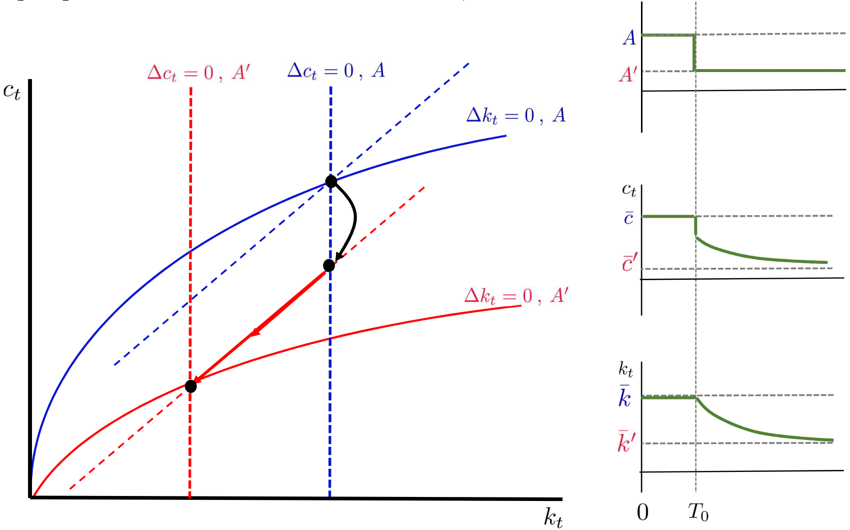
$$u'(c_t) = \beta u'(c_{t+1}) \left[A_t f'(k_{t+1}) + (1-\delta)k_{t+1}\right]$$

- $\Delta c_t = 0$ locus shifts

- $\Delta k_t = 0$ locus shifts

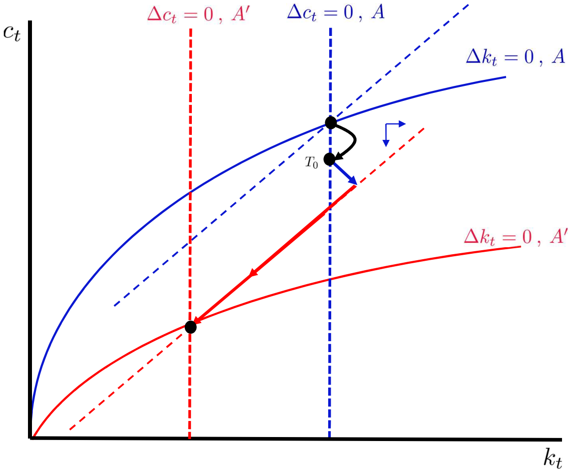
3. Dynamics - $\downarrow A$ - Unexpected - Permanent

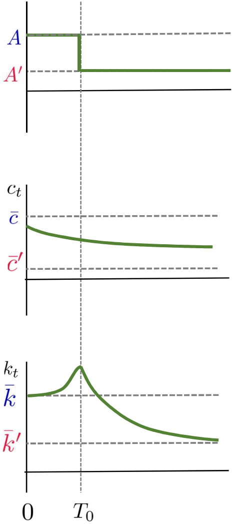
$\downarrow c_0$ and use up capital to smooth transition to lower $\bar{c}'$, $\bar{k}'$.



3. Dynamics - $\downarrow A$ - Expected - Permanent

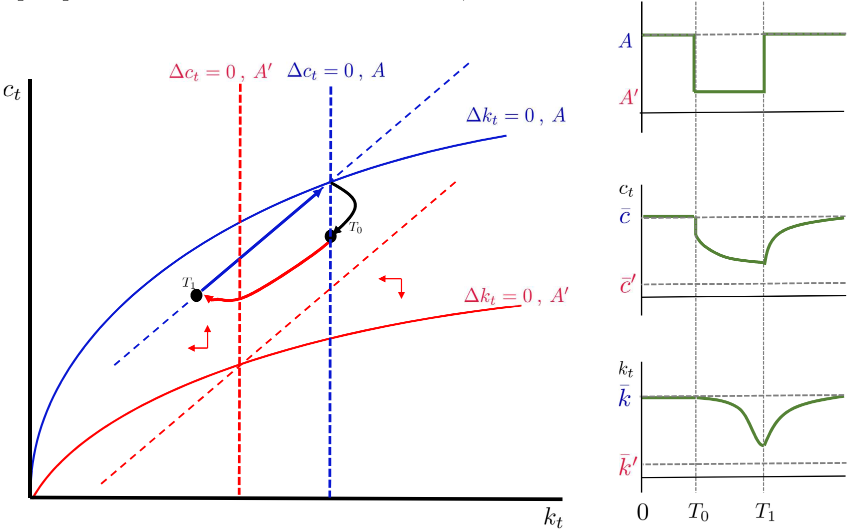
$\downarrow c_0$ and accumulate capital to smooth effect on consumption.





3. Dynamics - $\downarrow A$ - Unexpected - Transitory

$\downarrow c_0$ and use up capital to smooth transition to lower $\overline{c}'$, $\overline{k}'$.



3. Dynamics - $\downarrow A$ - Expected - Transitory

$\downarrow c_0$ and accumulate capital to smooth effect on consumption.