Honors - Economic Analysis III

Lecture 7: Dynamic Programming II

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This lecture

- Deriving the Euler equation - PS4
- Existence and uniqueness of value function
- Solving for the value function
- PS4 - Examples of writing out Bellman equations, solving for Euler equation
- Next:

  L8 Adding labor supply to the neoclassical model

  L9 RBC model

  L10 Asset pricing (Lucas, 1972)

Bellman equation

- Bellman equation - Writing $\mathbb{E}[f(z')|z] = \sum_{z'} \pi_z(z'|z) f(z')$

$$V(x,z) = \max_{x'} F(x,x',z) + \beta \mathbb{E}\left[V(x',z')|z\right]$$

subject to the constraint

$$x' \in \Gamma(x,z)$$

Bellman equation

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1. First order condition - Differentiate the Bellman equation for x'

$$F_2(x,x',z) + \beta \mathbb{E}\left[V_1(x',z')|z\right] = 0$$

Bellman equation

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2. Envelope condition - Differentiate the Bellman equation for x

$$V_1(x,z) = F_1(x,x',z)$$

Bellman equation

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2. Envelope condition - Differentiate the Bellman equation for x

$$V_1(x,z) = F_1(x,x',z)$$

- Combined $\rightarrow$ Euler equation

$$F_2(x,x',z) = \beta \mathbb{E}\left[F_1(x',x'',z')|z\right] \quad \rightarrow \quad x'(x,z)$$

Example - Neoclassical model

- Bellman equation - $(x, x', z) \rightarrow (k, k', a)$

$$V(k,a) = \max_{k'} u\Big(af(k) + (1-\delta)k - k'\Big) + \beta\mathbb{E}\left[V(k',a')|a\right]$$

subject to the constraint

$$k' \in [0, af(k) + (1-\delta)k]$$

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1. First order condition

$$-u'(c) + \beta\mathbb{E}[V_1(k',a')|a] = 0$$

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1. First order condition

$$-u'(c) + \beta \mathbb{E}[V_1(k', a')|a] = 0$$

2. Envelope condition

$$V_1(k, a) = u'(c)\left[af'(k) + (1-\delta)\right]$$

Example - Neoclassical model

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2. Envelope condition

$$V_1(k,a) = u'(c)\left[af'(k) + (1-\delta)\right]$$

- Combined $\to$ Euler equation

$$u'(c) = \beta \mathbb{E}[u'(c')\left[a'f'(k') + (1-\delta)\right]|a] \quad \to \quad k'(k,a)$$

Example - Cake eating w stochastic R

- Bellman equation

$$V(W,R) = \max_{W'} u(RW - W') + \beta \mathbb{E}\left[V(W',R')|R\right]$$

subject to the constraint

$$W' \in [0, RW]$$

1. First order condition

$$-u'(c) + \beta \mathbb{E}[V_W(W',R')|R] = 0$$

2. Envelope condition

$$V_W(W,R) = Ru'(c)$$

- Combined $\rightarrow$ Euler equation

$$u'(c) = \beta \mathbb{E}[R'u'(c')|R] \quad \rightarrow \quad W'(W,R)$$

What about the value function?

- This still doesn't solve our problems, we keep on differentiating $V(x, z)$ but ...
  - Does the value function $V(x, z)$ exist?
  - Is the value function unique?
  - Is the value function differentiable?
  - How do we compute the value function?

- Luckily for us, all of these questions are interrelated!

- Constructive proof of existence and uniqueness of $V(x, z)$ provides a recipe for how to compute it

- Stokey, Lucas and Prescott (1989) - *"Recursive Methods in Economic Dynamics"*

Fixed point

- Recall the Solow model with $\mathcal{T} : \mathbb{R}_+ \to \mathbb{R}_+$

$$k_{n+1} = \mathcal{T}(k_n) := (1-\delta)k_n + sf(k_n)$$

- Steady state was a *fixed point* k such that

$$k^* = \mathcal{T}(k^*)$$

- Suppose we start off at some arbitrary $k_0 > 0$ then

$$k_1 = \mathcal{T}(k_0)$$

$$k_2 = \mathcal{T}(k_1) = \mathcal{T}^2(k_0)$$

- Found that

$$\lim_{n\to\infty} |k_n - k_{n-1}| = \lim_{n\to\infty} \left|\mathcal{T}^n(k_0) - \mathcal{T}^{n-1}(k_0)\right| = 0$$

- Then

$$\lim_{n\to\infty} T^n(k_0) = k^*$$

Fixed point

- Imagine we had $\mathcal{T}$: Functions $\rightarrow$ Functions

$$V_{n+1} = \mathcal{T}(V_n) := \max_{x'} F(x, x', z) + \beta \mathbb{E}\left[V_n(x', z')|z\right]$$

- Value function is a *fixed point* V^* such that

$$V^* = \mathcal{T}(V^*)$$

- Suppose we start off at some arbitrary V_0 then

$$V_1 = \mathcal{T}(V_0)$$

$$V_2 = \mathcal{T}(V_1) = \mathcal{T}^2(V_0)$$

- And it was true that

$$\lim_{n\to\infty} ||V_n - V_{n-1}|| = \lim_{n\to\infty} \left|\left|\mathcal{T}^n(V_0) - \mathcal{T}^{n-1}(V_0)\right|\right| = 0$$

- Then

$$\lim_{n\to\infty} T^n(V_0) = V^*$$

Contraction mapping theorem - Part I

- Let $\mathcal{T}$ map bounded functions to bounded functions

$$\mathcal{T} : B(X) \to B(X) \quad , \quad f \in B(X) \quad , \quad f : X \to \mathbb{R}$$

- If $\mathcal{T}$ satisfies *Blackwell's sufficient conditions*:

  1. Monotonicity

  Let $f, g \in B(X)$ such that $f(x) > g(x)$, then

  $$\mathcal{T}f(x) > \mathcal{T}g(x)$$

  2. Discounting

  There exists some $\beta \in (0, 1)$ such that

  $$\mathcal{T}(f + a)(x) \leq \mathcal{T}f(x) + \beta a$$

- Then $\mathcal{T}$ is a *Contraction Mapping*

$$\sup_{x \in X} |\mathcal{T}f(x) - \mathcal{T}g(x)| \leq \beta \sup_{x \in X} |f(x) - g(x)|$$

Contraction mapping theorem - Part I

- Let $\mathcal{T}$ map bounded functions to bounded functions

$$\mathcal{T} : B(X) \to B(X) \quad , \quad V \in B(X) \quad , \quad V : X \to \mathbb{R}$$

- Suppose that $\mathcal{T}$ satisfies *Blackwell's sufficient conditions*:

  1. Monotonicity - Let $V(x, y) < V'(x, y)$ for all $(x, y) \in X$

$$\max_{x'} F(x, x', y) + \beta \mathbb{E}[V(x', y)] \leq \max_{x'} F(x, x', y) + \beta \mathbb{E}[V'(x', y)]$$

  2. Discounting

$$\mathcal{T}(V + a)(x, y) = \mathcal{T}V(x, y) + \beta a$$

- Then $\mathcal{T}$ is a *Contraction Mapping*

$$\sup_{(x,y)\in X} |\mathcal{T}V(x, y) - \mathcal{T}V'(x, y)| \leq \beta \sup_{(x,y)\in X} |V(x, y) - V'(x, y)|$$

Contraction mapping theorem - Part II

- If $\mathcal{T} : B(X) \to B(X)$ is a contraction mapping with modulus β:

1. $\mathcal{T}$ has a unique fixed point $V^* \in B(X)$ such that $\mathcal{T}V^* = V^*$

2. For any $V_0 \in B(X)$,

$$\sup_{(x,y)\in X} \left| \mathcal{T}^n V_0(x,y) - V^*(x,y) \right| \leq \beta^n \sup_{(x,y)\in X} \left| V_0(x,y) - V^*(x,y) \right|$$

for $n = 0, 1, 2, \ldots$

Contraction mapping theorem - Part II

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for $n = 0, 1, 2, \ldots$

- Let $S \subset B(X)$ be a closed set, then if $\mathcal{T}(S) \subset S$, then $V^* \in S$
  - If $\mathcal{T}$ maps continuous fns to continuous fns then V^* is continuous
  - If $\mathcal{T}$ maps increasing fns to increasing fns then V^* is increasing
  - If $\mathcal{T}$ maps concave fns to concave fns then V^* is concave

Example - Neoclassical model

- Bellman equation

$$V(k) = \max_{k'} u\Big(f(k) + (1-\delta)k - k'\Big) + \beta V(k)$$

subject to the constraint

$$k' \in [0, f(k) + (1-\delta)k]$$

1. First order condition

$$-u'(c) + \beta V_k(k')] = 0$$

2. Envelope condition

$$V(k) = u'(c)\,[f'(k) + (1-\delta)]$$

- Combined $\rightarrow$ Euler equation

$$u'(c) = \beta[z' f'(k') + (1-\delta)] \quad \rightarrow \quad k'(k)$$

Example - Neoclassical model

- Bellman equation

$$V(k) = \max_{k'} \, u\Big(f(k) + (1-\delta)k - k'\Big) + \beta V(k')$$

subject to the constraint

$$k' \in [0, f(k) + (1-\delta)k]$$

- Value function is bounded

$$\bar{k} = \arg\max_{k} f(k) - \delta k$$

$$V : [0, \bar{k}] \to \mathbb{R}_+ \quad , \quad V(k) \leq \frac{u(f(k^*) + (1-\delta)k^*)}{1-\beta} \quad , \quad V \in B([0, \bar{k}])$$

Example - Neoclassical model

- Bellman equation

$$V(k) = \max_{k'} u\Big(f(k) + (1-\delta)k - k'\Big) + \beta V(k')$$

subject to the constraint

$$k' \in [0, f(k) + (1-\delta)k]$$

- Monotonicity - Let $V < \tilde{V}$

$$\begin{aligned}
V(k) &= \max_{k'} u(f(k) + (1-\delta)k - k') + \beta V(k') \\
&= u(f(k) + (1-\delta)k - k^*_V(k)) + \beta V(k') \\
&\leq u(f(k) + (1-\delta)k - k^*_V(k)) + \beta \tilde{V}(k') \\
&\leq u(f(k) + (1-\delta)k - k^*_{\tilde{V}}(k)) + \beta \tilde{V}(k') \\
&= \max_{k'} u(f(k) + (1-\delta)k - k') + \beta \tilde{V}(k') \\
V(k) &\leq \tilde{V}(k)
\end{aligned}$$

Example - Neoclassical model

- Bellman equation

$$V(k) = \max_{k'} u\Big(f(k) + (1-\delta)k - k'\Big) + \beta V(k')$$

subject to the constraint

$$k' \in [0, f(k) + (1-\delta)k]$$

- Discounting

$$\begin{aligned}
\mathcal{T}(V+a)(k) &= \max_{k' \in [0, f(k)+(1-\delta)k]} u(f(k) + (1-\delta)k - k') + \beta[V(k') + a] \\
&= \max_{k' \in [0, f(k)+(1-\delta)k]} u(f(k) + (1-\delta)k - k') + \beta V(k') + \beta a \\
\mathcal{T}(V+a)(k) &\leq V(k) + \beta a
\end{aligned}$$

Example - Neoclassical model

- Special case: $u(c) = \log c$, $f(k) = k^{\alpha}$, $\delta = 1$
- Guess an initial value function $V_0(k') = 0$
- Solve optimization problem

$$V_1(k) = \max_{k'} \log\left(k^{\alpha} - k'\right) + \beta \times 0$$

- First order condition

$$k' = 0$$

- Substitute back

$$V_1(k) = \log(k^{\alpha}) = \alpha \log k$$

Example - Neoclassical model

- Special case: $u(c) = \log c$, $f(k) = k^\alpha$, $\delta = 1$
- With the updated value $V_1(k') = \alpha \log k'$
- Solve optimization problem

$$V_2(k) = \max_{k'} \log\left(k^\alpha - k'\right) + \beta\alpha \log k'$$

- First order condition

$$-\frac{1}{k^\alpha - k'} + \beta\alpha\frac{1}{k'} = 0 \quad \rightarrow \quad k' = \frac{\beta\alpha}{1+\beta\alpha}k^\alpha$$

- Substitute back

$$V_1(k) = \log\left(k^\alpha - \frac{\alpha\beta}{1+\alpha\beta}k^\alpha\right) + \beta\alpha \log\left(\frac{\beta\alpha}{1+\beta\alpha}k^\alpha\right)$$

$$V_1(k) = \alpha(1+\beta\alpha)\log k + \log\frac{1}{1+\beta\alpha} + \alpha\beta\log\frac{\beta\alpha}{1+\beta\alpha}$$

Example - Neoclassical model

- Special case: $u(c) = \log c$, $f(k) = k^\alpha$, $\delta = 1$
- With the updated value $V_1(k) = \alpha(1+\beta\alpha)\log k + \log \frac{1}{1+\beta\alpha} + \alpha\beta \log \frac{\alpha\beta}{1+\beta\alpha}$
- Solve optimization problem

$$V_2(k) = \max_{k'} \log\left(k^\alpha - k'\right) + \beta V_1(k')$$

- First order condition

$$-\frac{1}{k^\alpha - k'} + \beta\alpha(1+\beta\alpha)\frac{1}{k'} = 0 \quad \rightarrow \quad k' = \frac{\beta\alpha + (\beta\alpha)^2}{1+\beta\alpha+(\beta\alpha)^2}k^\alpha$$

- Substitute back

$$V_2(k) = \alpha(1+\beta\alpha+(\beta\alpha)^2)\log k + \beta \log\frac{1}{1+\beta\alpha} + \beta\alpha^2\log\frac{\beta\alpha}{1+\beta\alpha}$$

$$+\log\frac{1}{1+\beta\alpha+(\beta\alpha)^2} + (\beta\alpha+(\beta\alpha)^2)\log\frac{\beta\alpha+(\beta\alpha)^2}{1+\beta\alpha+(\beta\alpha)^2}$$

Example - Neoclassical model

- After n iterations

$$V_n(k) = \alpha(1 + \alpha\beta + (\alpha\beta)^2 + \cdots + (\alpha\beta)^n) \log k + Const.$$

- **CMT**: *Iterating forward as $n \to \infty$ we know that $V_n(k) \to V(k)$*

$$V(k) = \frac{\alpha}{1 - \alpha\beta} \log k + Const.$$

- Now can solve for the policy function

$$V(k) = \max_{k'} \log \left(k^\alpha - k'\right) + \beta \frac{\alpha}{1 - \alpha\beta} \log k' + \beta\, Const.$$

- First order condition

$$\frac{1}{k^\alpha - k'} = \frac{\alpha\beta}{1 - \alpha\beta} \frac{1}{k'} \quad \to \quad k'(k) = \alpha\beta k^\alpha \quad , \quad c(k) = (1 - \alpha\beta)k^\alpha$$

- Recognize the solution: $\log k_{t+1} - \log \bar{k} = \alpha \left[\log k_t - \log \bar{k}\right] \to \widehat{k}_{t+1} = \alpha \widehat{k}_t$

Computational receipe - 'Algorithm'

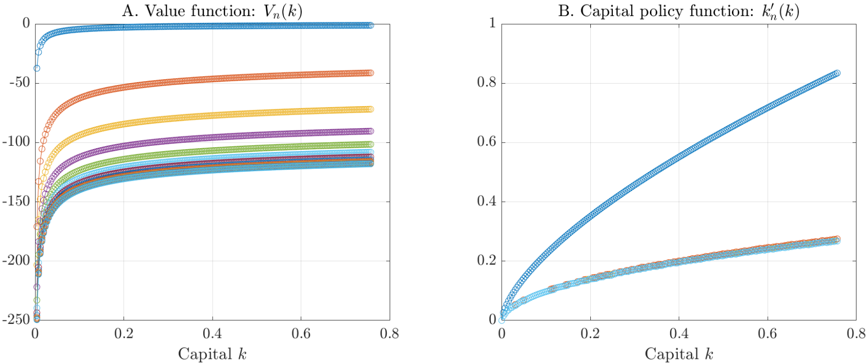
Algorithm

- Guess V_0
- Solve for $k'(k)$ given V_0
- Compute V_1
- Check if $\max|V_1 - V_0| < \varepsilon$
- If so, done.
- If not, update $V_0 = V_1$

Example

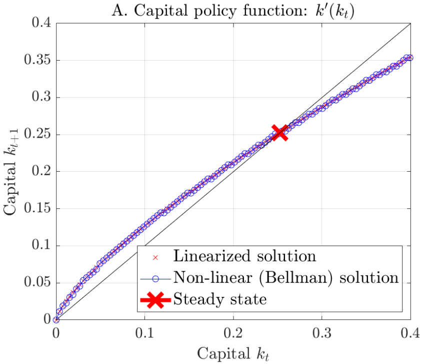
- Set up a grid for $k \in \{0, k_1, \ldots, k_N\}$ where $k_N = 3.5 \times \bar{k}$
- Restrict choices of k' to lie on the grid
- Starting guess $V_0(k) = 0$
- Solve for $V(k)$ and $k'(k)$
- Compare solution to the approximate linear solution that we found using our earlier approach: $\widehat{k}' = \lambda_1 \widehat{k}$.

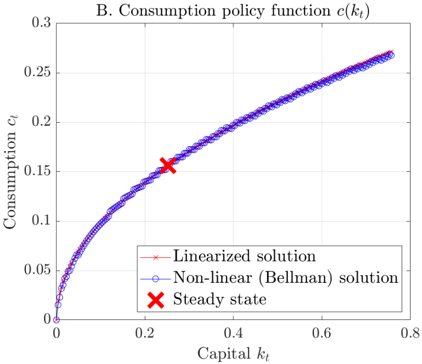
Value function and policy function



$$V_n(k) = \max_{k' \in \Gamma(k)} u(c) + \beta V_{n-1}(k) \qquad \rightarrow \qquad k'_n(k)$$

2. Dynamics - Local dynamics





Homework - Cake eating

- An individual has a cake of size x. Each period they choose how much of the cake to consume. Any cake that is not consumed grows at rate $R > 1$ between periods. The individual has utility function $u(c) = \log c$ and discounts the future at rate $\beta < 1$.
- Write down the sequence problem
- Write down the Bellman equation $V(s)$ where s is the state vector
- Use the FOC(s) and envelope condition to derive the Euler equation
- Using a starting guess of $V_0(s) = 0$, solve for the optimal policy and use this to iterate backward to $V_1(s)$ and $V_2(s)$
- Use this to establish an expression for $V_n(s)$
- Show that V satisfies the sufficient conditions such that $\mathcal{T}V$ is a contraction (write out $\mathcal{T}$).
- What can we then say about $\lim_{n\to\infty} V_n(s)$?
- Use this to derive the policy function $W'(s)$

END